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Integral Equations and Operator Theory

Syndetically Hypercyclic Operators

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Abstract. Given a continuous linear operator $T \in L(X)$ defined on a separable $\mathcal{F}$-space X, we will show that T satisfies the Hypercyclicity Criterion if and only if for any strictly increasing sequence of positive integers $\{n_k\}_k$ such that $\sup_k\{n_{k+1} - n_k\} < \infty$, the sequence $\{T^{n_k}\}_k$ is hypercyclic. In contrast we will also prove that, for any hypercyclic vector $x \in X$ of T, there exists a strictly increasing sequence $\{n_k\}_k$ such that $\sup_k\{n_{k+1} - n_k\} = 2$ and $\{T^{n_k}x\}_k$ is somewhere dense, but not dense in X. That is, T and $\{T^{n_k}\}_k$ do not share the same hypercyclic vectors.

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1. Introduction

Let X be a separable $\mathcal{F}$-space (i.e., metrizable and complete topological vector space). In this paper we work with continuous linear operators $T : X \to X$, referred to simply as operators. A sequence of operators $\{T_n\}_n$ is said to be a *hypercyclic sequence* on X if there exists some $x \in X$ such that its orbit

$$Orb(\{T_n\}_n, x) := \{x, T_1x, T_2x, ...\}$$

is dense in X. In this case the vector x is called *hypercyclic* for the sequence $\{T_n\}_n$. An operator T is *hypercyclic* on X if $\{T^n\}_n$ is a hypercyclic sequence of operators. Note that if $\{T_n\}_n$ is a hypercyclic sequence of operators on X, then X is necessarily separable.

Being hypercyclic, for a single operator as well as for a sequence of commuting operators with dense range, is equivalent (see, for instance [9, Theorem 1 and Proposition 1]) to a property called *topological transitivity*: A sequence of continuous maps $\{T_n\}_n$ on a topological space X is *transitive* if for any pair U, V of non

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empty open subsets of X there is a positive integer n_0 such that

$$T_{n_0}(U) \cap V \neq \emptyset$$

A single map $T : X \to X$ is *transitive* if the sequence $\{T^n\}_n$ is transitive.

A sufficient condition for hypercyclicity, the well known Hypercyclicity Criterion, independently discovered by Kitai [10] and Gethner and Shapiro [8], has been the fundamental tool for proving hypercyclicity. The following version of the hypercyclicity criterion was given by Bès (see [4]).

Theorem 1.1. *(The Hypercyclicity Criterion) Let T be an operator on a separable $\mathcal{F}$-space X. Suppose that there exists a strictly increasing sequence of positive integers $\{n_k\}_k$ for which there are*

1. *a dense subset $X_0 \subset A$ such that $T^{n_k}x \to 0$ for every $x \in X_0$, and*
2. *a dense subset $Y_0 \subset X$ and a sequence of mappings $\{S_{n_k} : Y_0 \to X\}_k$ such that*
   a) $S_{n_k}y \to 0$ for every $y \in Y_0$,
   b) $T^{n_k}S_{n_k}y \to y$ for every $y \in Y_0$.

Then T is hypercyclic on X.

T is said to satisfy the *Hypercyclicity Criterion* if it satisfies the hypothesis of last theorem. Every example of hypercyclic operator in the literature so far seems to satisfy the Hypercyclicity Criterion, but it is still an open question if every hypercyclic operator satisfies it.

Definition 1.2. A strictly increasing sequence of positive integers $\{n_k\}_k$ is said to be *syndetic* if $\sup_k\{n_{k+1} - n_k\} < \infty$ (see, e.g., [7]). An operator T on X is called *syndetically hypercyclic* if for any syndetic sequence of positive integers $\{n_k\}_k$, the sequence $\{T^{n_k} : X \to X\}_k$ is hypercyclic.

We will show that $T \in L(X)$ is syndetically hypercyclic if and only if T satisfies the Hypercyclicity Criterion. This partially settles a question posed by Bès (personal communication), who asked if every hypercyclic operator is syndetically hypercyclic (see also [11]). Bès's problem was motivated by a result of Ansari which asserts that the sequence $\{T^{pn}\}_n$ is hypercyclic for each $p \in \mathbb{N}$ whenever T is hypercyclic [1] (see also [2, Theorem 2.5]). By our equivalence, an affirmative answer to Bès's question would prove that every hypercyclic operator satisfies the Hypercyclicity Criterion.

In the final section we show that, for any hypercyclic operator $T \in L(X)$ on a general locally convex space X, and for any vector x hypercyclic for T, there exists a strictly increasing sequence of positive integers such that $\sup_k\{n_{k+1} - n_k\} = 2$ and $\{T^{n_k}x\}_k$ is not dense in X. However, the sequence $\{T^{n_k}x\}_k$ turns out to be somewhere dense. This disproves Proposition 4.1 in [4] and establishes a difference between sub-orbits and orbits of vectors under T: If for some $x \in X$ the orbit

$$Orb(T,x) := \{x, Tx, T^2x, ...\}$$

is somewhere dense, then $Orb(T,x)$ is everywhere dense (and then T is a hypercyclic operator on X). This remarkable result was recently proved by Bourdon and Feldman [5].

2. Syndetically hypercyclic operators and the Hypercyclicity Criterion

The key result of this section, which is fundamental for the desired equivalence with the Hypercyclicity Criterion, remains valid for continuous maps on topological spaces. For this reason we keep the first part of the section within this general context.

Definition 2.1. Let X be a topological space and let $T : X \longrightarrow X$ be a continuous map. T is called *weakly mixing* if $T \times T : X \times X \longrightarrow X \times X$ is topologically transitive.

The implication '(i) implies (ii)' in the following proposition is due to Furstenberg [6, Prop. II.11]. We include its proof for the sake of completeness.

Proposition 2.2. *Let $T : X \longrightarrow X$ be a continuous map on a topological space X. Then the following are equivalent:*

(i) T is weakly mixing.

(ii) For any pair of non-empty open subsets $U, V \subseteq X$, and for any strictly increasing sequence $\{n_k\}_k$ with $\sup_k\{n_{k+1} - n_k\} < \infty$, there exists k_0 such that $T^{n_{k_0}}U \cap V \neq \emptyset$.

(iii) It suffices in (ii) to consider only those sequences $\{n_k\}_k$ for which there is some $m \geq 1$ with $n_{k+1} - n_k \in \{m, 2m\}$ for all k.

Proof. (i) implies (ii) [Furstenberg]: Given $\{n_k\}_k$ and U, V satisfying the hypothesis of (ii), we set $m := \sup_k\{n_{k+1} - n_k\}$, and the m-product map

$$\underbrace{T \times T \times \cdots \times T}_{m-times} : X \times X \times \cdots \times X \longrightarrow X \times X \times \cdots \times X,$$

is transitive [6, Prop. II.3]. Then, there is an $n \in \mathbb{N}$ such that

$$T^nU \cap (T^i)^{-1}V \neq \emptyset$$

for all $i = 1, \ldots, m$. This implies that $T^{n+i}U \cap V \neq \emptyset$ for all $i = 1, \ldots, m$. By the assumption on $\{n_k\}_k$, we have that $\{n_k : k \in \mathbb{N}\} \cap \{n+1, \ldots, n+m\} \neq \emptyset$. If we select n_{k_0} in this intersection we get $T^{n_{k_0}}U \cap V \neq \emptyset$.
(ii) implies (iii) is trivial.

(iii) implies (i): We will show that, given non-empty open subsets $U, V_1, V_2 \subset X$, there is an $n \in \mathbb{N}$, such that

$$T^n U \cap V_i \neq \emptyset,$$

for $i = 1, 2$. This will imply that T is weakly mixing (see [3, Lemma 5]).

Fix $m \in \mathbb{N}$ such that $T^m V_1 \cap V_2 \neq \emptyset$ (Such m exists if (iii) is satisfied). By continuity, we can find $\tilde{V_1} \subset V_1$ open and non-empty such that $T^m \tilde{V_1} \subset V_2$. Assumption (iii) implies the existence of some $l \in \mathbb{N}$ such that $T^{l+j} U \cap \tilde{V_1} \neq \emptyset$, for $j = 0, m$. Otherwise we would find a strictly increasing sequence of positive integers $\{n_k\}_k$ such that $n_{k+1} - n_k \in \{m, 2m\}$, and $T^{n_k} U \cap \tilde{V_1} = \emptyset$ for all $k \in \mathbb{N}$. We then have

$$T^{l+m} U \cap \tilde{V_1} \neq \emptyset,$$

$$T^{l+m} U \cap T^m \tilde{V_1} \supset T^m (T^l U \cap \tilde{V_1}) \neq \emptyset.$$

If we fix $n := l + m$, we conclude

$$T^n U \cap V_1 \neq \emptyset,$$

$$T^n U \cap V_2 \neq \emptyset.$$

□

We notice that condition (ii) can be equivalently formulated as: *For any pair of non-empty open subsets $U, V \subset X$, and for any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $T^{n+i} U \cap V \neq \emptyset$, $i = 0, \ldots, m$.* That is, the set of integers $N(U, V) := \{n \in \mathbb{N} \,:\, T^n U \cap V \neq \emptyset\}$ is *replete*.

In [4, Theorem 2.3] Bès and Peris showed that an operator $T \in L(X)$ on a separable $\mathcal{F}$-space X satisfies the Hypercyclicity Criterion if and only if T is weakly mixing. Combining this result with the previous proposition we obtain:

Theorem 2.3. *Let $T : X \longrightarrow X$ be an operator on a separable $\mathcal{F}$-space X. Then the following are equivalent:*

(i) T satisfies the Hypercyclicity Criterion.

(ii) T is syndetically hypercyclic.

The observation after Proposition 2.2 now yields the following equivalence with the Hypercyclicity Criterion: Let X be a separable $\mathcal{F}$-space, $\mathcal{U}$ the family of all non-empty open subsets of X, and $T \in L(X)$. Then T satisfies the Hypercyclicity Criterion if and only if

$$\forall U, V \in \mathcal{U} \ \forall m \in \mathbb{N} \ \exists n \in \mathbb{N} : \quad T^i U \cap V \neq \emptyset, \quad i = n, \ldots, n + m.$$

We recall that if $\{n_k\}_k$ is such that $\sup_k \{n_{k+1} - n_k\} = \infty$, then there are hypercyclic weighted shift operators T on l_2 such that $\{T^{n_k}\}_k$ is not a hypercyclic sequence of operators [11].

3. Different Sets of Hypercyclic Vectors

In this section we will show that, even in the case when an operator T satisfies the Hypercyclicity Criterion and $\{n_k\}_k$ is a syndetic sequence, T and $\{T^{n_k}\}_k$ do not have to share the same sets of hypercyclic vectors, i.e., in some cases the set of hypercyclic vectors for $\{T^{n_k}\}_k$ is strictly contained in the set of hypercyclic vectors for T. More precisely, we will prove that if T is a hypercyclic operator and $x \in X$ is any hypercyclic vector for T, there exists a syndetic sequence $\{n_k\}_k$, such that the orbit

$$Orb(\{T^{n_k}\}_k, x) := \{x, T^{n_1}x, \dots\}$$

is somewhere dense but not everywhere dense. As we mentioned in the introduction, this establishes a difference between the full orbit and the sub-orbit associated to a sequence $\{n_k\}_k$, for a single operator T, which should be compared with the result of Bourdon and Feldman [5].

We will also give a result for standard dynamical systems that came out of a conversation with L. Frerick.

Lemma 3.1. *Let X be a topological space without isolated points and let $T : X \to X$ be a continuous map. If $x \in X$ satisfies that $Orb(T, x)$ is dense in X, then, for any syndetic sequence $\{n_k\}_k$ of positive integers, the associated orbit $Orb(\{T^{n_k}\}_k, x)$ is somewhere dense.*

Proof. If $\{n_k\}_k$ is syndetic, we set $m := \sup_k\{n_{k+1} - n_k\}$. Without loss of generality $n_1 > m$. Since X has no isolated points and $Orb(T, x)$ is dense in X, we have

$$X = \overline{\{T^n x \ : \ n \geq n_1 - m\}} = \bigcup_{i=0}^{m} \overline{\{T^{n_k - i}x \ : \ k \in \mathbb{N}\}}.$$

We define $M_i := \overline{\{T^{n_k-i}x \ : \ k \in \mathbb{N}\}}$, $i = 0, \dots, m$. If $int(M_0) \neq \emptyset$, then we are done. If not $X = \bigcup_{i=1}^{m} M_i$, and this would imply

$$X = \overline{T(X)} = \bigcup_{i=1}^{m} \overline{T(M_i)} = \bigcup_{i=0}^{m-1} M_i = \bigcup_{i=1}^{m-1} M_i.$$

By iterating this process we would arrive at $X = M_1$, thus $X = \overline{T(M_1)} = M_0$, which is a contradiction. □

Our main result in this section holds for general locally convex spaces X.

Theorem 3.2. *Let T be a hypercyclic operator on a locally convex space X and let $x \in X$ be a hypercyclic vector for T. Then there exists a sequence of positive integers $\{n_k\}_k$ with $\sup_k\{n_{k+1} - n_k\} = 2$ such that $\{T^{n_k}x\}_k$ is somewhere dense but not everywhere dense.*

Proof. Let $x \in X$ be a hypercyclic vector for T. Then $Orb(T, x)$ is linearly independent. Therefore, $T^2x \notin span\{x, Tx, T^3x, T^4x\}$. Then there exists an element x^* in the dual X' of X such that $\langle x^*, T^2x\rangle = 1$ and $\langle x^*, T^ix\rangle = 0$ for $i = 0, 1, 3, 4$. Let $P : X \longrightarrow \mathbb{K}^3$ be given by

$$P(y) = (\langle x^*, y\rangle, \langle x^*, Ty\rangle, \langle x^*, T^2y\rangle)$$

for all $y \in X$. P is linear, continuous and, since by definition $P(x) = (0, 0, 1)$, $P(Tx) = (0, 1, 0)$ and $P(T^2x) = (1, 0, 0)$, we have that P is surjective. We define

$$n_k = \begin{cases} k & if \quad |\langle x^*, T^{k+1}x\rangle| > |\langle x^*, T^{k+2}x\rangle| \\ k+1 & otherwise. \end{cases}$$

(i) If $|\langle x^*, T^{k+1}x\rangle| > |\langle x^*, T^{k+2}\rangle|$, then

$$P(T^{n_k}x) = P(T^kx) = (\langle x^*, T^kx\rangle, \langle x^*, T^{k+1}x\rangle, \langle x^*, T^{k+2}x\rangle).$$

(ii) If $|\langle x^*, T^{k+1}x\rangle| \leq |\langle x^*, T^{k+2}x\rangle|$, then

$$P(T^{n_k}x) = P(T^{k+1}x) = (\langle x^*, T^{k+1}x\rangle, \langle x^*, T^{k+2}x\rangle, \langle x^*, T^{k+3}x\rangle).$$

In consequence, for any $k \in \mathbb{N}$, the second coordinate of $P(T^{n_k}x)$ has magnitude greater than or equal to the first or the third coordinate of $P(T^{n_k}x)$. By continuity these inequalities pass on to anything in the closure of the set $\{P(T^{n_k}x)\}_k$. In particular $(1, 0, 1) \notin \overline{\{P(T^{n_k}x) : k \in \mathbb{N}\}}$. The surjectivity of P implies that $\{T^{n_k}x\}_k$ can not be dense. □

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