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Chaotic polynomials on Banach spaces

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Abstract

We show the existence of chaotic (in the sense of Devaney) polynomials on Banach spaces of q-summable sequences. Such polynomials P consist of composition of the backward shift with a certain fixed polynomial p of one complex variable on each coordinate. In general we also prove that P is chaotic in the sense of Auslander and Yorke if and only if 0 belongs to the Julia set of p.

Keywords: Hypercyclic vectors; Polynomial operators; Chaos; Julia sets

1. Introduction and preliminaries

A continuous map $f : X \to X$ on a metric space X is *topologically transitive* if, for each pair $U, V \subset X$ of non-empty open sets, there is $n \in \mathbb{N}$ so that $f^n(U) \cap V \neq \emptyset$. For complete, separable and perfect (i.e., without isolated points) spaces X, transitivity is equivalent to the existence of $x \in X$ such that the orbit

$$\mathrm{Orb}(x) := \{x, fx, f^2x, \ldots\}$$

is dense in X. The point x is then called *hypercyclic* and the set of hypercyclic points is a dense G_δ-subset of X.

The map f is called *chaotic* in the sense of Auslander and Yorke (from now on, *AY-chaotic*) if it is transitive and it has sensitive dependence on initial conditions, that is

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$$\exists\varepsilon > 0\ \forall x \in X\ \forall\delta > 0\ \exists y \in X\ \exists n \in \mathbb{N}:$$
$$d(x, y) < \delta \quad \text{but} \quad d(f^n x, f^n y) > \epsilon.$$

The map f is chaotic in the sense of Devaney (from now on, *D-chaotic*) if it is transitive, the set $\mathcal{P}$ of periodic points of f is dense in X, and f has sensitive dependence on initial conditions. Several authors (see, e.g., [1]) showed that sensitive dependence on initial conditions is redundant in Devaney's definition.

We will be concerned with chaos for polynomials on (separable) complex Banach spaces X.

A map $Q : X \to X$ is an *m-homogeneous polynomial* if there exists $A : X \times X \times \cdots \times X \to X$ multilinear and continuous such that $Q(x) = A(x, x, \ldots, x)$ for all $x \in X$. $P : X \to X$ is a (continuous) *polynomial* if $P = \sum_{m=0}^{l} Q_m$, where each Q_m is an m-homogeneous polynomial.

In this context transitivity is equivalent to the existence of dense orbits, that is, P is hypercyclic.

Many examples of hypercyclic linear operators (in other words, hypercyclic 1-homogeneous polynomials) on Banach spaces are known. See, e.g., the survey of Grosse-Erdmann [3]. However, in contrast to this fact, Bernardes showed that there are no hypercyclic m-homogeneous polynomials of degree $m > 1$ on any Banach space [2]. Motivated by this result, Aron (personal communication) asked whether there exist hypercyclic non-homogeneous polynomials of degree strictly greater than 1 on Banach spaces. Here we solve Aron's question affirmatively by showing that there are even D-chaotic polynomials of degree greater than 1 on Banach spaces. In [6] it was proved that there are chaotic homogeneous polynomials of degree $m > 1$ on Fréchet spaces. More examples of hypercyclic and chaotic polynomials on Banach and Fréchet spaces are given in [5].

Our polynomials will be defined on spaces X of q-summable sequences ($1 \leqslant q < \infty$)

$$l_q := \left\{ x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} / \|x\|_q := \left(\sum_{i=1}^{\infty} |x_i|^q \right)^{1/q} < \infty \right\},$$

and on the space of null sequences

$$c_0 := \left\{ x = (x_i)_i \in \mathbb{C}^{\mathbb{N}} / \lim_i x_i = 0,\ \|x\|_\infty := \sup_i |x_i| \right\}.$$

We first show that the polynomial of degree m,

$$P : X \to X, \quad x = (x_1, x_2, \ldots) \mapsto \big((x_2 + 1)^m - 1, (x_3 + 1)^m - 1, \ldots \big)$$

is D-chaotic on $X = l_q$ or c_0, for any $m > 1$, thus answering Aron's question.

This motivates us to study general polynomials of the form

$$P : X \to X, \quad x = (x_1, x_2, \ldots) \mapsto \big(p(x_2), p(x_3), \ldots \big),$$

where $p : \mathbb{C} \to \mathbb{C}$ is a fixed complex polynomial. We characterize AY-chaos of P and show that it is equivalent to the fact that 0 is a fixed point of p belonging to the Julia set of p. Finally we prove that P is D-chaotic if 0 is a repelling fixed point of p.

2. First examples

A classical result of Rolewicz [7] establishes that, for $X = l_q$ or c_0, the weighted backward shift operator

$$\lambda B : X \to X, \quad x = (x_1, x_2, \ldots) \mapsto (\lambda x_2, \lambda x_3, \ldots)$$

is hypercyclic if $|\lambda| > 1$. Actually, it is D-chaotic.

By using the operator of Rolewicz and an argument with commutative diagrams, we give the first examples of chaotic polynomials of degree greater than 1 defined on a Banach space.

The following result, which is well known, can be found in, e.g., [4].

Lemma 2.1. *Let X be a Banach space and f, g, and ϕ be continuous maps defined on X with values in X such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"'] \\ X \arrow[r, "g"] & X \end{tikzcd}$$

is commutative (*i.e.*, $\phi \circ f = g \circ \phi$), *and ϕ has a dense range. If f is D-chaotic, then g is D-chaotic.*

In this section we will consider the polynomials of degree m,

$$P : X \to X, \quad (x_i)_i \mapsto \big((x_{i+1} + 1)^m - 1\big)_i,$$

where $X = l_q$ or c_0 $(1 \leqslant q < \infty)$ and $m \geqslant 2$. It is clear that $P = \sum_{k=1}^{m} Q_k$ with $Q_k((x_i)_i) := \binom{m}{k}(x_{i+1}^k)_i$ a k-homogeneous continuous polynomial for each k. Therefore P is also well defined and continuous.

Proposition 2.2. *P is D-chaotic.*

Proof. We define $\phi : X \to X$, $\phi((x_i)_i) := (e^{x_i} - 1)_i$, which is locally Lipschitz (hence continuous), and has dense range. We have, for $\lambda := m$ and $x \in X$, that

$$(\phi \circ \lambda B)x = \phi\big((\lambda x_{i+1})_i\big) = (e^{\lambda x_{i+1}} - 1)_i = (P \circ \phi)x.$$

That is, the diagram

$$\begin{tikzcd} X \arrow[r, "\lambda B"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"'] \\ X \arrow[r, "P"] & X \end{tikzcd}$$

is commutative. By Lemma 2.1 we conclude that P is D-chaotic. $\square$

3. Chaotic polynomials and Julia sets

Inspired by the examples of the previous section, we will consider now the general polynomials

$$P : X \to X, \quad (x_i)_i \mapsto \big(p(x_{i+1})\big)_i,$$

where $p : \mathbb{C} \to \mathbb{C}$ is a complex polynomial with $p(0) = 0$, a necessary condition in order that P is well defined.

A natural question is whether the (chaotic) dynamics of the polynomial P in infinite dimensions and the (chaotic) dynamics of the polynomial p of one complex variable are related. We will show that the answer is positive.

We first recall some basic facts and definitions from complex dynamics.

A family $\mathcal{A}$ of meromorphic functions defined on an open set $D \subset \bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is *normal* if it is equicontinuous on any compact subset of D. A sufficient condition for normality is given by

Montel's theorem. *If there are three values that are omitted by every $f \in \mathcal{A}$, then $\mathcal{A}$ is normal.*

Given a polynomial $p : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ ($p(\infty) := \infty$), its associated *Fatou set* is

$$\mathcal{F}(p) := \big\{z \in \bar{\mathbb{C}} / \mathcal{A} := \{p^n,\ n \in \mathbb{N}\} \text{ is normal on some neighbourhood of } z\big\}.$$

The *Julia set* is $\mathcal{J}(p) := \bar{\mathbb{C}} \setminus \mathcal{F}(p)$.

A periodic point z of p with period k is *repelling* if $|(p^k)'(z)| > 1$.

The following equality (due to Fatou and Julia) relates the Julia set to the set of repelling periodic points:

$$\mathcal{J}(p) = \overline{\{z \in \mathbb{C} / z \text{ is a repelling periodic point of } p\}}.$$

It is well known that, for polynomials of degree greater than 1, the Julia set $\mathcal{J}(p)$ is a non-empty p-invariant compact set such that the restriction $p : \mathcal{J}(p) \to \mathcal{J}(p)$ is D-chaotic.

We also need the following lemma, whose proof is included for the sake of completeness.

Lemma 3.1. *Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with* $\deg(p) \geqslant 2$. *Given an element $x_0 \in \mathbb{C}$ in the Julia set of p, a neighbourhood $U \subset \mathbb{C}$ of x_0, $\varepsilon > 0$, and a finite collection $\{z_1, \ldots, z_n\}$ of elements in $\mathbb{C}$, then there are $x_i \in U$, $i = 1, \ldots, n$, and $m \in \mathbb{N}$ such that $|p^m(x_i) - z_i| < \varepsilon$, $i = 1, \ldots, n$.*

Proof. Let u be a repelling periodic point of p in U. If k is the period of u, since $|(p^k)'(u)| > 1$, there is a disk U_0 centered at u contained in U such that $p^k(U_0) \supset U_0$. Then

$$U_j := p^{jk}(U_0) \subset p^{(j+1)k}(U_0) = U_{j+1}, \quad j \in \mathbb{N}.$$

By Montel's theorem the family $\mathcal{A} := \{p^{jk} : U_0 \to \bar{\mathbb{C}} / j \in \mathbb{N}\}$ has, at most, two exceptional points which are omitted (actually, at most one since ∞ is omitted) by $\mathcal{A}$. Therefore we find $\{y_1, \dots, y_n\} \subset \mathbb{C}$ with $|y_i - z_i| < \varepsilon$ such that y_i is not omitted by $\mathcal{A}$, $i = 1, \dots, n$. That is,

$$\{y_1, \dots, y_n\} \subset \bigcup_{j\in\mathbb{N}} p^{jk}(U_0) = \bigcup_{j\in\mathbb{N}} U_j.$$

In particular, since the U_j's are increasing, there is $j' \in \mathbb{N}$ satisfying $y_i \in U_{j'} \subset p^m(U)$, $i = 1, \dots, n$, where $m := j'k$. $\square$

Theorem 3.2. *Let $X = l_q$ or c_0 and let $P : X \to X$ be the continuous polynomial given by $P((x_i)_i) := (p(x_{i+1}))_i$, where $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree strictly greater than* 1 *such that $p(0) = 0$. The following conditions are equivalent*:

(i) *P is AY-chaotic,*
(ii) *P is hypercyclic,*
(iii) *P has sensitive dependence on initial conditions,*
(iv) 0 *belongs to the Julia set of p.*

Proof. (i) $\Rightarrow$ (ii) It follows from the definition of AY-chaotic.

(ii) $\Rightarrow$ (iii) For $\varepsilon := 1$, given $x \in X$ and $\delta > 0$, we find $k \in \mathbb{N}$ such that $\|x - (x_1, x_2, \dots, x_k, 0, 0, \dots)\| < \delta$. There is $z \in X$ hypercyclic for P so that $\|x - z\| < \delta$. Since the orbit of $\bar{x} := (x_1, \dots, x_k, 0, \dots)$ tends to 0, by the hypercyclicity of z we get $n \in \mathbb{N}$ satisfying $\|P^n\bar{x} - P^n z\| > 2$. That is, either $\|P^n\bar{x} - P^n x\| > \varepsilon$ or $\|P^n z - P^n x\| > \varepsilon$.

(iii) $\Rightarrow$ (iv) If $0 \notin \mathcal{J}(p)$ then $\{p^n,\ n \in \mathbb{N}\}$ is normal on some neighbourhood of 0. This implies that the sequence of derivatives $\{(p^n)',\ n \in \mathbb{N}\}$ is also normal on some neighbourhood of 0. In particular, since 0 is a fixed point of p, we have that $\{(p^n)',\ n \in \mathbb{N}\}$ is uniformly bounded on some neighbourhood of 0, that is, there exist $\delta, M > 0$ such that $|(p^n)'(z)| \leqslant M$ for all $n \in \mathbb{N}$ and for each $z \in \mathbb{C}$ such that $|z| < \delta$. Therefore,

$$|p^n z| \leqslant M|z|, \quad \forall n \in \mathbb{N},\ \forall z\colon\ |z| < \delta.$$

By definition of P we get

$$\|P^n x\| \leqslant M\|x\|, \quad \forall n \in \mathbb{N},\ \forall x\colon\ \|x\| < \delta.$$

But this means that P does not have sensitive dependence on initial conditions at 0, which is a contradiction.

(iv) $\Rightarrow$ (i) Since we already know that (ii) implies (iii) and that (ii) + (iii) = (i), we just need to prove that (iv) implies (ii). By Lemma 3.1, for any finite collection $\{x_1, \dots, x_m\} \subset \mathbb{C}$ and any $\delta > 0$, there are $\{z_1, \dots, z_m\} \subset \mathbb{C}$ and $n \in \mathbb{N}$ such that $|z_i| < \delta$ and $|p^n z_i - x_i| < \delta$, $i = 1, \dots, n$. We are done if we show that P is transitive, i.e., if for any $x, y \in X$ and $\varepsilon > 0$, there are $n \in \mathbb{N}$ and $\bar{y} \in X$ such that

$$\|\bar{y} - y\| < \varepsilon \quad \text{and} \quad \|P^n\bar{y} - x\| < \varepsilon.$$

Let $m \in \mathbb{N}$ be such that

$$\bigl\|x - (x_1, \dots, x_m, 0, \dots)\bigr\| < \frac{\varepsilon}{2} \quad \text{and} \quad \bigl\|y - (y_1, \dots, y_m, 0, \dots)\bigr\| < \frac{\varepsilon}{2}.$$

Pick $\delta := \varepsilon/2m$ and find $n > m$, $\{z_1, \ldots, z_m\} \subset \mathbb{C}$ with $|z_i| < \delta$ and $|p^n z_i - x_i| < \delta$, $i = 1, \ldots, m$. Define then

$$\bar{y} := \big(y_1, \ldots, y_m, 0, \ldots, 0, \overset{n+1}{z_1}, \ldots, z_m, 0, \ldots\big).$$

We conclude $\|\bar{y} - y\| < \varepsilon$ and

$$\|P^n \bar{y} - x\| < \frac{\varepsilon}{2} + \big\|(p^n z_1, \ldots, p^n z_m, 0, \ldots) - (x_1, \ldots, x_m, 0, \ldots)\big\| < \varepsilon. \qquad \square$$

We would like to compare Theorem 3.2 with Proposition 2.2. The linear part of the polynomial of Proposition 2.2 is the operator mB, which is D-chaotic. In general, polynomial dynamics is richer than linear dynamics in the sense that the linear part of an AY-chaotic polynomial might be non-chaotic. More precisely, if we consider the complex polynomial $p(z) := z + z^2$, it is clear that 0 is a non-repelling fixed point of p which belongs to the Julia set of p. Therefore, the corresponding polynomial P defined on X is AY-chaotic. On the other hand, the linear part of P is the backward shift B, which is neither hypercyclic nor has sensitive dependence on initial conditions!

Chaos in the sense of Devaney seems to be a stronger condition than AY-chaos, within our framework.

Proposition 3.3. *Let $X = l_q$ and let $P : X \to X$ be the continuous polynomial given by $P((x_i)_i) := (p(x_{i+1}))_i$, where $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree strictly greater than* 1 *such that $p(0) = 0$. If* 0 *is repelling then P is D-chaotic.*

Proof. Since repelling points are contained in the Julia set, in view of Theorem 3.2 we just have to show the density of periodic points. Fix $x \in X$ and $\varepsilon > 0$ and select $|p'(0)| > \lambda > 1$. We pick $m \in \mathbb{N}$ with $\|x - (x_1, \ldots, x_m, 0, \ldots)\| < \varepsilon/2$ and $\delta < \varepsilon/6m$ such that $p(U_0) \supset \lambda U_0$ for any disk U_0 centered at 0 of radius smaller than δ. By Lemma 3.1, we find $n > m$ and $\{z_{1,1}, \ldots, z_{1,m}\} \subset \mathbb{C}$ such that $|z_{1,i}| < \delta$ and $|p^n z_{1,i} - x_i| < \delta$, $i = 1, \ldots, m$. Without loss of generality n is chosen so that $\sum_{k=1}^{\infty} \lambda^{-kn} < 1$. Proceeding by induction we select $\{z_{j,i} \in \mathbb{C},\ j > 1,\ i = 1, \ldots, m\}$ satisfying $|z_{j+1,i}| < \lambda^{-jn}\delta$ and $p^n z_{j+1,i} = z_{j,i}$, $j \in \mathbb{N}$, $i = 1, \ldots, m$. We then define

$$z := \big(z_{0,1}, \ldots, z_{0,m}, 0, \ldots, 0, \overset{n+1}{z_{1,1}}, \ldots, z_{1,m}, 0, \ldots, 0, \overset{2n+1}{z_{2,1}}, \ldots\big),$$

where $z_{0,i} := p^n z_{1,i}$, $i = 1, \ldots, m$. By definition z is a periodic point of P, and taking $\bar{x} := (x_1, \ldots, x_m, 0, \ldots)$, we have

$$\|x - z\| \leqslant \|x - \bar{x}\| + \|\bar{x} - z\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{3} + \left(\sum_{k=1}^{\infty} \lambda^{-kn}\right)\frac{\varepsilon}{6} < \varepsilon. \qquad \square$$

Remark 3.4. When $X = c_0$, a stronger result can be obtained. Namely, all conditions in Theorem 3.2 are equivalent to D-chaos of P. For such a purpose we need to construct periodic points as in the proof of the previous proposition, but this time we only need that $(z_{j,i})$ tends to 0 when j tends to infinity, which can be done if we just assume that 0 belongs to the Julia set of p.

We do not know for $X = l_q$ if $0 \in \mathcal{J}(p)$ suffices to get that P is D-chaotic, but we think that this is not enough.

Conjecture 3.5. *If $X = l_q$, $1 \leqslant q < \infty$, then $P : X \to X$ is D-chaotic if and only if 0 is a repelling fixed point of p.*

References

[1] J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, Amer. Math. Monthly 99 (1992) 332–334.
[2] N. Bernardes, On orbits of polynomial maps in Banach spaces, Quaestiones Math. 21 (1998) 311–318.
[3] K.G. Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. 36 (1999) 345–381.
[4] F. Martínez-Giménez, A. Peris, Chaos for backward shift operators, Internat. J. Bifur. Chaos 12 (2002) 1703–1715.
[5] F. Martínez-Giménez, A. Peris, Hypercyclic and chaotic polynomials, in preparation.
[6] A. Peris, Erratum to: "Chaotic polynomials on Fréchet spaces" [Proc. Amer. Math. Soc. 127 (1999) 3601–3603], Proc. Amer. Math. Soc. 129 (2001) 3759–3760.
[7] S. Rolewicz, On orbits of elements, Studia Math. 33 (1969) 17–22.