Double Sampling Hotelling's T^2 Charts

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Abstract

Two double sampling T^2 charts are discussed. They only differ in how the second sample is used to suggest to the practitioner the state of the process. An optimal method using a genetic algorithm is given for designing these charts based on the average run length of the (ARL). An analytical method is used to determine run length performance of the chart. Comparisons are made with various other control charting procedures. Some recommendations are given.

Key words: Adaptive control chart, ARL , MEWMA, run length distribution, statistical process control, genetic algorithm.

1 Introduction

The double sampling $\bar{X}$ chart was introduced into the literature by Croasdale [1] and later modified by Daudin [2]. Irianto and Shinozaki [3], Carot, Jabaloyes, and Carot [4], He [5], Hee and Grigoryan [6], and He, Grigoryan, and Singh [7] examine the design of double sampling as well as charts based on triple sampling.

A double sampling chart makes use of the one sample and possibly a second sample at each sampling stage to make a judgement about the state of the process. If it is judged that the information in the first sample is not enough to make an inference about the quality of the process, then the items of the second sample are measured and coupled with the first sample to aid the practitioner in making a decision. This idea (when practical) can be extended to more than two samples. It is important to note that the total number of items sampled at a given sampling stages is restricted by the output of the process.

Various other adaptive control charts have been introduced into the literature. In the univariate case, variable sampling interval (VSI) have been investigated, among others, by Reynolds, Amin, Arnold, and Nachlas [8] and Runger and Pignatiello [9]. Prabhu, Montgomery, and Runger [10] evaluated the performance of charts that have variable sampling intervals as well as variable sample sizes. Aparisi [11] investigates the use of variable sampling sizes (VSS) with the Hotelling's T^2 chart. The use of variable sampling intervals (VSI) for the T^2 control chart was studied by Aparisi and Haro [12]. In a later work, Aparisi and Haro [13] compared the performance of VSS-VSI Hotelling's T^2 chart with the VSS Hotelling's T^2 chart, VSI Hotelling's T^2 chart, and the MEWMA chart.

He [5] developed a multivariate double sampling $\bar{X}$ chart. Champ and Aparisi [14] studied two double sampling control charts used to monitor the mean vector μ of a $p \times 1$ vector $\mathbf{X}$ of quality measurements. One the double sampling T^2 chart and the other a double sampling combined T^2 chart. They used an analytical method for obtaining the run length properties of these charts and a genetic algorithm approach to select the charts' parameters. Reynolds and Kim [15] studied the multiple sampling T^2 chart as a special case of the multiple sampling multivariate EWMA chart in which the multiple sampling T^2 chart studied by He and Grigoryan [16] is a special case. The double sampling T^2 chart is a special case of the multiple sampling T^2 chart. Both Reynolds and Kim [15] and He and Grigoryan [16] used simulation to study the average run length (ARL) properties of the chart whereas Champ and Aparisi [14] present a analytical method for obtaining the ARL of the chart. Champ and Aparisi [14] and He and Grigoryan [16] both used a genetic algorithm approach to select the optimal chart parameter. In this article, we will present the method of Champ and Aparisi [14] for determining the ARL of the chart. Also we express two forms of the chart. The one the practitioner would use and an equivalent form of the chart that is useful in determining the run length properties of the chart. The chart parameter for both forms are the same allowing these to be selected without knowledge of the in-control mean vector and covariance matrix that will be provided by the

practitioner.

As are most multivariate statistical methods, these charts are designed under the assumption that $\mathbf{X}$ has a multivariate normal distribution with a $p \times 1$ mean vector μ and positive definite covariance matrix Σ . The process is considered to be in-control if $\mu = \mu_0$ and $\Sigma = \Sigma_0$. The out-of-control process that we consider is when $\mu \neq \mu_0$ and $\Sigma = \Sigma_0$. Information about the quality of the process at each sampling stage will be in the form of two independent random samples (the first of size n_1 and the second of size n_2) taken together periodically from the output of the process. We let $\mathbf{X}_{i,j}$ represent the vector of quality measurements to be taken on the j th item ($j = 1, 2, \dots, n_i$) from the i th sample ($i = 1, 2$).

Each of the two double sampling schemes first plots the Hotelling's T^2 statistic

$$T_{k,1}^2 = n_1 (\bar{\mathbf{X}}_{k,1} - \mu_0)^T \Sigma_0^{-1} (\bar{\mathbf{X}}_{k,1} - \mu_0)$$

versus the sample number k , where $\bar{\mathbf{X}}_{k,1}$ is the mean of the first sample collected at sampling stage k . The observed value of $T_{k,1}^2$ is used to suggest one of the following to the practitioner:

- (1) Signal the process is potentially out-of-control if $T_{k,1}^2 \geq h_1$ ($h_1 > 0$).
- (2) If $0 < T_{k,1}^2 < w_1 < h_1$, wait until the next sampling stage to study the process further.
- (3) If $w_1 \leq T_{k,1}^2 < h_1$, measure the items in the second sample and pool this information with the first to make a decision about the state of the process.

How the information is used in the combined sample to make a suggestion to the practitioner about the state of the process is how our two proposed charts differ. The first of our two charts summarizes the information in the combined sample using the statistic

$$Q_k^2 = \frac{1}{n} (n_1 T_{k,1}^2 + n_2 T_{k,2}^2),$$

with

$$T_{k,2}^2 = n_2 (\bar{\mathbf{X}}_{k,2} - \mu_0)^T \Sigma_0^{-1} (\bar{\mathbf{X}}_{k,2} - \mu_0),$$

where $\bar{\mathbf{X}}_{k,2}$ is the mean vector of the second sample and $n = n_1 + n_2$. This chart signals a potential out-of-control process if $Q_k^2 \geq h$; otherwise recommends no action be taken by the practitioner. Since the statistic Q_k^2 is a weighted average of Hotelling's T^2 statistics, we refer to this chart as the double sampling (DS)

combined T^2 chart. The second of these charts summarizes the data in the combined sample using the statistic

$$T_k^2 = n (\bar{\mathbf{X}}_k - \mu_0)^T \Sigma_0^{-1} (\bar{\mathbf{X}}_k - \mu_0) \quad (1)$$

where

$$\bar{\mathbf{X}}_k = \frac{1}{n} (n_1 \bar{\mathbf{X}}_{k,1} + n_2 \bar{\mathbf{X}}_{k,2}) \quad (2)$$

is a weighted average of the sample mean vectors. This chart signals if $T_k^2 \geq h$; otherwise no further action at this sampling stage is recommended. We refer to this chart as the *DS* T^2 chart.

When using the Hotelling's T^2 chart (Hotelling [17]), one question that arises after the chart has signalled is "what component(s) of the mean vector are the cause of the signal?" Various authors have addressed this problem. Blazek, Novic and Scott [18], Iglewicz and Hoaglin [19], Fuchs and Benjamin [20], Subramanyam and Houshmand [21], and Atienza, Ching and Wah [22] give graphical methods for determining the cause of the signal. Hawkins [23] and Wade and Wodall [24] use adjusted regression of the individual variables of the T^2 statistic to interpret the cause of the signal. Runger et al. [25] propose the use of different metric distances. A method similar to discriminant analysis was proposed by Murphy [26]. The approach developed by Mason, Tracy, and Young [27, 28] analyzes the factors resulting from the decomposition of the Hotelling's T^2 statistic, whose value indicates a probable out-of-control situation. Recently, Aparisi, Avendaño and Sanz [29] have studied the use of neural networks for determining the cause of the signal. All these methods can be applied to both the *DS* combined T^2 and the *DS* T^2 charts.

It is our purpose in this article to evaluate the performance of the *DS* combined T^2 and the *DS* T^2 charts and to compare these charts with selected multivariate charts. Equivalent forms of the chart and distributional results that are useful in evaluating the performance of the chart are given in the next section. In Section 3, these charts are compared with Hotelling's [17] T^2 chart, the multivariate exponentially weighted moving average (MEWMA) of Lowry, et al [30], and the variable sample size (VSS) Hotelling's T^2 studied by Aparisi [11]. An example is given in the fourth section which is followed by a concluding section. We have also included in the Appendix a derivation of some distributional results for those who may be interested.

2 Evaluating the Run Length Distribution

The average run length (ARL) of the chart is a common measure of how well a chart performs in detecting an out-of-control process. The run length is the number of the sampling stage at which the chart first signals. We examine the run length distribution for both the DS combined and $DS T^2$ charts when μ_0 and Σ_0 will be given. In this case, the run length distributions of both charts are geometric each with a parameter of the form $1 - P_a$, where P_a is the probability the chart does not signal at any given sampling stage. The ARL of the chart (as are other parameters such as the standard deviation of the run length and percentage points of the run length distribution) is functionally related to the probability P_a . We have that

$$ARL = \frac{1}{1 - P_a}.$$

As will be seen, the probability P_a is a function of the distributional parameters μ , μ_0 , and Σ_0 of the distribution of the vector of quality measurements, $\mathbf{X}$, only through the value d , where $d^2 = (\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)$ with μ the out-of-control value of the mean vector provided $\Sigma = \Sigma_0$. Also, P_a is a function of the sample sizes n_1 and n_2 . Further, we show how the run length distribution depends on the chart parameters w_1 , h_1 , and h . We could state this explicitly by writing

$$P_a = P_a(d, n_1, n_2, w_1, h_1, h),$$

but these function arguments will be implicit in what follows. Note that the process is in-control when $d = 0$ and $\Sigma = \Sigma_0$. While it will not be shown in this paper, it should also be noted that run length distribution also depends on changes in the covariance matrix.

For the DS combined T^2 chart which is based on the statistics $T_{k,1}^2$ and Q_k^2 at sampling stage k , the probability the chart does not signal at time $k = 1$ is given by

$$P_a = P_{a,1} + P_{a,2}$$

where $P_{a,1} = P[T_{1,1}^2 < w_1]$ and $P_{a,2} = P[w_1 \leq T_{1,1}^2 < h_1, Q_1^2 < h]$. It is shown in Graybill [33] that $T_{1,i}^2$ has a non-central chi square distribution with p degrees of freedom and noncentrality parameter λ_i , where $\lambda_i = n_i d^2$ for $i = 1, 2$. Thus, we have

$$P_{a,1} = P[T_{1,1}^2 < w_1] = P[\chi_{p,\lambda_1}^2 < w_1] = F_{\chi_{p,\lambda_1}^2}(w_1)$$

and

$$\begin{aligned} P_{a,2} &= P [Q_1^2 < h \mid w_1 \leq T_{1,1}^2 < h_1] P [w_1 \leq T_{1,1}^2 < h_1] = \int_{w_1}^{h_1} F_{T_{k,2}^2} \left(\frac{nh - n_1 y}{n_2} \right) f_{T_{k,1}^2}(y) dy \\ &= \int_{w_1}^{h_1} F_{\chi_{p,\lambda_2}^2} \left(\frac{nh - n_1 y}{n_2} \right) f_{\chi_{p,\lambda_1}^2}(y) dy, \end{aligned}$$

for all $k = 1, 2, 3, \dots$, where $f_{T_{k,1}^2}$ and f_{χ_{p,λ_1}^2} are the probability density functions of the random variables $T_{k,1}^2$ and χ_{p,λ_1}^2 , respectively, and $F_{T_{k,2}^2}$ and F_{χ_{p,λ_2}^2} are the cumulative distribution function of the random variables $T_{k,2}^2$ and χ_{p,λ_2}^2 , respectively. Here χ_{p,λ_i}^2 is a random variable with a noncentral chi square distribution with p degrees of freedom and noncentrality parameter $\lambda_i = n_i d^2$ ($i = 1, 2$). As can be seen, the probability $P_{a,1}$ and the probability $P_{a,2}$ depend on the distributional parameters μ , μ_0 , and Σ_0 only through the parameter d .

The second of these charts, the $DS\ T^2$ chart, does not signal if $0 < T_{k,1}^2 < w_1$ or if $w_1 \leq T_{k,1}^2 < h_1$ and $0 < T_k^2 < h$. Using our previous results, we have for this chart that $P_{a,1} = F_{\chi_{p,\lambda_1}^2}(w_1)$. The value of $P_{a,2}$ when the process is in-control is given by

$$P_{a,2} = \int_{w_1}^{h_1} F_{\chi_{p,(n_1/n_2)v}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p,0}^2}(v) dv \quad (3)$$

When the process is out-of-control ($d > 0$), then we have

$$\begin{aligned} P_{a,2} &= \int_{-\sqrt{w_1}-\sqrt{n_1}d}^{-\sqrt{w_1}-\sqrt{n_1}d} \int_0^{h_1-(z+\sqrt{n_1}d)^2} F_{\chi_{p,\theta}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p-1,0}^2}(v) \phi(z) dv dz \\ &\quad + \int_{-\sqrt{w_1}-\sqrt{n_1}d}^{\sqrt{w_1}-\sqrt{n_1}d} \int_{w_1-(z+\sqrt{n_1}d)^2}^{h_1-(z+\sqrt{n_1}d)^2} F_{\chi_{p,\theta}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p-1,0}^2}(v) \phi(z) dv dz \\ &\quad + \int_{\sqrt{w_1}-\sqrt{n_1}d}^{\sqrt{h_1}-\sqrt{n_1}d} \int_0^{h_1-(z+\sqrt{n_1}d)^2} F_{\chi_{p,\theta}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p-1,0}^2}(v) \phi(z) dv dz \end{aligned} \quad (4)$$

where ϕ is the density function of a standard normal distribution and

$$\theta = \frac{n_1}{n_2} \left[\left(z + \frac{n}{\sqrt{n_1}} d \right)^2 + v \right]$$

The derivation of Equations (3) and (4) are given in the Appendix. We can see from Equation (4) that the probabilities $P_{a,1}$ and $P_{a,2}$ depend on the distributional parameters μ , μ_0 , and Σ_0 only through the parameter d . Hence, the run length distributions and in particular the ARLs for both of these charts depend only on the distributional parameters through the parameter d .

One measure of cost in using these charts is the number of sampled items on which the vector of quality measurements are taken. At each sampling stage, it is easy to see that the expected or average sample size

is given by

$$n_0 = n_1 + n_2 P(w_1 \leq T_{1,1}^2 < h_1) = n_1 + n_2 \left(F_{\chi_{p,\lambda_1}^2}(h_1) - F_{\chi_{p,\lambda_1}^2}(w_1) \right).$$

μ_0 and $\Sigma = \Sigma_0$

3 Chart Design and Comparisons

In choosing a double sampling T^2 , the practitioner must select the charting parameters n_1 , n_2 , w_1 , h_1 , and h . One well known way of selecting these charting parameters is based on the *ARL* criteria of having a fixed in-control *ARL*, say ARL_0 , and a minimum out-of-control *ARL* for a specified shift in the process. The criteria that we will use is a modification of this method. Firstly, we add the requirement that the sample sizes are to be selected such that $n_1 < n_0 < n_2$, where n_0 is the average sample size, $E(n)$. Secondly, we require a bound $n_{\max}$ on the total number $n_1 + n_2$ of items selected at each sampling stage. A chart that is selected based on these criteria we will refer to as an optimal double sampling chart. This design method requires that the practitioner specify the in-control *ARL*, ARL_0 , the magnitude d of the process shift one desires to detect, the number of quality measurements p , and the average sample size n_0 , and a bound $n_{\max}$ on the total items sampled. The restriction $E(n) = n_0$ is used to make comparisons with the other charts the practitioner may choose. In addition, the value n_0 provides the user with an indication of the resources needed to collect the data.

Formally, the optimization problem is as follows: Given the length (d) of the process shift that is desirable to detect, the number of variables (p), the desired in-control *ARL* (ARL_0), maximum total sample size ($n_{\max}$) and the average sample size desired when the process is in-control (n_0), find sample sizes n_1 and n_2 , warning limit (w_1), control limits h_1 and h that minimizes $ARL(d)$ subject to $w_1 < h_1$, $E(n) = n_0$, $n_1 < n_0 < n_2$, and $n_1 + n_2 \leq n_{\max}$. In order to solve this optimization problem a program has been written that first makes a search using a genetic algorithm to obtain a solution. The solution is then refined using a local search technique. This method is used by Aparisi and García-Díaz (2004) for optimization of EWMA and MEWMA control charts. Another program is available from the second author that determines the *ARL* values for both the *DS* combined T^2 and *DS* T^2 charts for a given set of values p , h , h_1 , w_1 , n_1 , n_2 , and d . Both programs are available from the second author.

Chart parameters for the *DS* combined T^2 and *DS* T^2 charts are presented in Tables 1-9. Each chart is

designed to have an in-control ARL of 400. We consider charts with shift magnitudes of 0.5, 1.0, 1.5, and 2.0, number of quality measurements $p = 3, 5$ and 10, and $n_0 = 3, 6$ and 10. For each of these cases, the parameters of the optimal charts are given. Also included in these tables, for comparison purposes, are the Hotelling's T^2 chart, Lowry, et al. [30] MEWMA chart, and VSS Hotelling's T^2 (see Aparisi [11]) each of which has been designed to have a minimum out-of-control $ARLs$ for the given shift.

[Insert Tables 1-9 about here.]

A study has also been carried out for the case in which the in-control ARL is 1000. Similar results were obtained and are available from the authors.

Examining Tables 1-9 some general conclusions can be made. It can be seen that the $DS T^2$ always outperforms the DS combined T^2 chart. Thus, we recommend the $DS T^2$ over the DS combined T^2 .

Hotelling's T^2 control chart is quite simple in comparison with the $DS T^2$. However, what we lose in simplicity we gain in ARL performance. For small shifts ($d = 0.5$ and 1) the ARL improvements for $DS T^2$ chart are very important, especially for $d = 0.5$. In some cases the out-of-control ARL obtained is about an eleventh of the ARL of the Hotelling's T^2 chart. However, for larger shifts the difference tends to be less. With large shifts ($d = 2$) and large samples sizes ($n_0 = 10$) the performance are essentially equivalent.

Also it is interesting to compare the ARL results against the MEWMA and VSS T^2 control charts. The performance of the MEWMA chart is the best for a small shift magnitudes of $d = 0.5$, with improvements against the $DS T^2$ chart in the range of 4% to 50%. The performance of the $DS T^2$ chart is always better than the MEWMA chart for the rest of shifts magnitudes studied. The better improvements are obtained for $d = 1$, with improvements in the range of 2% to 43%. Finally, the performance of the VSS T^2 control chart is always worse than the $DS T^2$ control chart. The differences in this case are small for $n_0 = 10$. However, when $n_0 = 3$ an improvement of up to 52% can be obtained.

4 An Illustrative Example

Aparisi et al.[32] gave an example of a process that produces connecting rods for automobile engines. Figure 1 illustrates three of the quality measurements taken on each rod, where L is the distance between centers with ϕ_1 and ϕ_2 the diameters of the connection to the piston and crankshaft, respectively. Supposing that, when process is in control, the means vector and the covariance matrix are both known and with the following

values:

$$\mu_0 = \begin{bmatrix} \mu_L \\ \mu_{\phi_1} \\ \mu_{\phi_2} \end{bmatrix} = \begin{bmatrix} 20 \\ 7 \\ 4 \end{bmatrix}; \Sigma_0 = \begin{bmatrix} 0.04 & 0.02 & 0.01 \\ 0.02 & 0.02 & 0.011 \\ 0.1 & 0.011 & 0.01 \end{bmatrix}$$

The standard Hotelling's T^2 control chart was employed to monitor for a change in the three quality measurements ($p = 3$) with sample size of 3. It is desired to use a $DS T^2$ that is optimal for detecting a shift in the mean vector with $d = \sqrt{(\mu - \mu_0)^T \Sigma_0^{-1} (\mu - \mu_0)} = 1$, an in-control $ARL = 400$, and an average sample size $n_0 = 3$. We find in Table 1 that the optimum parameters for this chart are: $h = 13.41$, $h_1 = 14.86$, $w = 7.50$, $n_1 = 2$, and $n_2 = 18$. The chart has an out-of-control ARL of 5.110 for $d = 1$. In comparison, the Hotelling's T^2 control chart has an out-of-control ARL of 20.781 (about 4 times more) and the out-of-control ARL for MEWMA chart is 5.550 (9% more).

[Insert Figure 1 here.]

Figure 2 shows a plot of the T^2 statistics verses the sample number k for five samples. Table 10 shows the mean vectors obtained at each sample stage. We note that at each sampling stage we take two samples, one of size $n_1 = 2$ and a second of size $n_2 = 18$. The monitoring begins by plotting the point $(1, T_{1,1}^2)$ statistic produced by the first sample of size $n_1 = 2$. This is the first point on the chart. As this point plots below the warning line, no further action is taken at sampling stage 1. That is, the second sample taken at sampling stage 1 of size $n_2 = 18$ is not measured. The second plotted point, $(2, T_{2,1}^2)$, also plots below the warning line. However at sampling stage $k = 3$, the observed value of $T_{3,1}^2$ plots between the warning line ($w_1 = 7.50$) and the control limit ($h_1 = 14.86$). This point is labelled $3a$ on the chart. The vector of quality measurements is taken on the second sample of size $n_2 = 18$ collected at sampling stage 3. After combining both $\bar{\mathbf{X}}$ statistics, using Equation (2), the resulting T^2 statistic (3b) calculated using Equation (1) has a value less than h . Thus, the evidence found in the samples at sampling stage 3 is not consider to indicate a potential out-of-control process. At sampling stage 4, the chart statistic $T_{4,1}^2$ plots (point 4) below the warning limit. The chart requires no further action by the practitioner. In the next sample stage, the chart statistic $T_{5,1}^2$ plots above the control limit h_1 causing the chart to signal a potential out-of-control process.

[Insert Figure 2 here.]

[Insert Table 10 here]

5 Conclusions

Double sampling is a simple addition to a control chart that significantly increases the ability of the chart in detecting various changes in the process. This is the case with the two double sampling control charts based on Hotelling's T^2 statistics introduced in this paper. They differ only in how they use the second sample to make a decision about the process. One, the *DS* combined T^2 control chart, uses a weighted average of the T^2 statistics associated with each sample. The other, the *DS* T^2 control chart, uses the Hotelling's T^2 statistic based on the pooled sample. Expressions were derived for obtaining the run length distribution of each chart. These were useful in implementing the criterion for obtaining an optimal double sampling chart.

It was demonstrated that the *ARL* performance of the *DS* T^2 chart is better than the *DS* combined T^2 chart when the in-control *ARL* is 400. Thus, of the two charts, we recommend the *DS* T^2 chart. An *ARL* comparison of the *DS* T^2 chart against the standard Hotelling's T^2 control chart shows that the improvements of *ARL* are very important for small to moderate size shifts. In addition, the *DS* T^2 control chart shows smaller *ARLs* for all shift magnitudes in comparison against the Variable Sample Size T^2 chart. However, the multivariate exponentially weighted moving average (MEWMA) control chart outperforms the *DS* T^2 control chart for very small shifts (Mahalanobis' distance equals to 0.5). Hence, it is a chart to be considered by practitioners.

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7 Appendix

The random variable T^2 can be written as

$$T^2 = \frac{n_2}{n} \left[\mathbf{Z}_2 + \frac{\sqrt{n_1}}{\sqrt{n_2}} \left(\mathbf{Z}_1 + \frac{n}{\sqrt{n_1}} \delta \right) \right]^T \left[\mathbf{Z}_2 + \frac{\sqrt{n_1}}{\sqrt{n_2}} \left(\mathbf{Z}_1 + \frac{n}{\sqrt{n_1}} \delta \right) \right]$$

where $\mathbf{Z}_i = \sqrt{n_i} \mathbf{P}_0^{-1} (\bar{\mathbf{X}}_i - \mu) \sim N_p(\mathbf{0}, \mathbf{I})$ and $\delta = \mathbf{P}_0^{-1} (\mu - \mu_0)$ with $\Sigma_0 = \mathbf{P}_0 \mathbf{P}_0^T$. The matrix $\mathbf{P}_0$ is the product of the matrix of normalized eigenvectors of the (positive definite matrix) Σ_0 and the diagonal matrix of the square roots of the associated eigenvalues. Note that for convenience, we are suppressing the sampling stage number k and in what follows unless stated otherwise when k is suppressed we take $k = 1$. Further it is easy to show that T_i^2 can be written as

$$T_i^2 = (\mathbf{Z}_i + \sqrt{n_i} \delta)^T (\mathbf{Z}_i + \sqrt{n_i} \delta)$$

and T_i^2 has a noncentral chi square distribution with p degrees of freedom and noncentrality parameter $\lambda_i = n_i d^2$ with $d^2 = \delta^T \delta$. When the process is in-control and $k = 1$, we have that $T_i^2 = \mathbf{Z}_i^T \mathbf{Z}_i$ has a central chi square distribution with p degrees of freedom. In what follows, we take $k = 1$ unless otherwise stated.

For the case in which the process is in-control, we then have

$$T^2 = \frac{n_2}{n} \left(\mathbf{Z}_2 + \frac{\sqrt{n_1}}{\sqrt{n_2}} \mathbf{Z}_1 \right)^T \left(\mathbf{Z}_2 + \frac{\sqrt{n_1}}{\sqrt{n_2}} \mathbf{Z}_1 \right).$$

Using Theorem 4.2.1 in Graybill [33], it follows that the conditional distribution of T^2 given $\mathbf{Z}_1$ is (n_2/n) times a noncentral chi square distribution with p degrees of freedom and noncentrality parameter $Y = (n_1/n_2) \mathbf{Z}_1^T \mathbf{Z}_1 = (n_1/n_2) T_1^2$. Thus, the distribution of $T^2 | T_1^2$ has the same distribution as $T^2 | \mathbf{Z}_1$. Further, we that the random variable $(n_1/n_2) T_1^2$ is distributed as (n_1/n_2) times a central chi square random variable with p degrees of freedom. It follows that when the process is in-control

$$\begin{aligned} P_{a,2} &= P [T^2 < h | w_1 \leq T_1^2 < h_1] P [w_1 \leq T_1^2 < h_1] \\ &= P \left[T^2 < h \left| \frac{n_1}{n_2} w_1 \leq \frac{n_1}{n_2} T_1^2 < \frac{n_1}{n_2} h_1 \right. \right] P \left[\frac{n_1}{n_2} w_1 \leq \frac{n_1}{n_2} T_1^2 < \frac{n_1}{n_2} h_1 \right] \\ &= \int_{n_1 w_1 / n_2}^{n_1 h_1 / n_2} F_{T^2 | (n_1/n_2) T_1^2} (h) f_{T_1^2} (r) dr = \int_{w_1}^{h_1} F_{\chi_{p, (n_1/n_2) v}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p,0}^2} (v) dv \end{aligned}$$

When the process is out-of-control, then the distribution of $T^2 | \mathbf{Z}_1$ is a noncentral chi square distribution with p degrees of freedom and noncentrality parameter

$$Y = \frac{n_1}{n_2} T_1^2 + 2\sqrt{n_1} U + (2n_1 + n_2) d^2$$

where $U = \delta^T \mathbf{Z}_1$ and $d^2 = \delta^T \delta$. It is now easy to see that when the process is out-of-control ($d > 0$) $T^2 | \mathbf{Z}_1$ has the same distribution as $T^2 | U, T_1^2$.

For the case in which process is out-of-control ($d > 0$), consider an orthogonal transformation $\mathbf{B}$ such

that $\mathbf{B}\delta = [d, 0, \dots, 0]^T$. We can write

$$T^2 = \frac{n_2}{n} \left[\mathbf{Z}_2^* + \frac{\sqrt{n_1}}{\sqrt{n_2}} \left(\mathbf{Z}_1^* + \frac{n}{\sqrt{n_1}} \delta^* \right) \right]^T \left[\mathbf{Z}_2^* + \frac{\sqrt{n_1}}{\sqrt{n_2}} \left(\mathbf{Z}_1^* + \frac{n}{\sqrt{n_1}} \delta^* \right) \right]$$

Since $\mathbf{Z}_1^* = \mathbf{B}\mathbf{Z}_1$ and $\mathbf{Z}_2^* = \mathbf{B}\mathbf{Z}_2$ are independent and identically distributed as $N_p(\mathbf{0}, \mathbf{I})$ random vectors, then the conditional distribution of T^2 given $\mathbf{Z}_1^* = \mathbf{B}\mathbf{Z}_1$ has a noncentral chi square distribution and noncentrality parameter Y^* given by

$$Y^* = \frac{n_1}{n_2} T_1^2 + 2\sqrt{n_1} d Z_{1,1}^* + (2n_1 + n_2) d^2,$$

where $Z_{1,1}^*$ is the first component of $\mathbf{Z}_1^*$.

We now observe that T_1^2 can now be express as

$$T_1^2 = (Z_{1,1}^* + \sqrt{n_1} d)^2 + \sum_{i=2}^p (Z_{i,1}^*)^2 = (Z + \sqrt{n_1} d)^2 + V,$$

where $Z_{i,1}^*$ representing the i th component of the random vector $\mathbf{Z}_1^*$, $Z = Z_{1,1}^*$, and $V = \sum_{i=2}^p (Z_{i,1}^*)^2$. It is easy to see that $V \sim \chi_{p-1,0}^2$ and $Z \sim N(0, 1)$ are independent. We observe that $T^2 | \mathbf{Z}_1^*$, $T^2 | Z, T_1^2$, and $T^2 | Z, V$ have the same distribution. The value of $P_{a,2}$ when the process is out-of-control is then determined as follows:

$$\begin{aligned} P_{a,2} &= P \left[T^2 < h \mid w_1 \leq V + (U/d + \sqrt{n_1} d)^2 < h_1 \right] P \left[w_1 \leq V + (U/d + \sqrt{n_1} d)^2 < h_1 \right] \\ &= \iint_{\mathbf{A}} F_{T^2|U,V} (h | u, v) f_V(v) \phi(z) dv dz, \end{aligned} \quad (5)$$

where $\mathbf{A} = \left\{ (z, v) \mid w_1 \leq (z + \sqrt{n_1} d)^2 + v < h_1, v > 0 \right\}$ and $\phi(z)$ is the density of a standard normal distribution. It now follows that Equation 5 can be expressed as

$$\begin{aligned} P_{a,2} &= \int_{-\sqrt{w_1}-\sqrt{n_1}d}^{-\sqrt{w_1}-\sqrt{n_1}d} \int_0^{h_1-(z+\sqrt{n_1}d)^2} F_{\chi_{p,\theta}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p-1,0}^2}(v) \phi(z) dv dz \\ &+ \int_{-\sqrt{w_1}-\sqrt{n_1}d}^{\sqrt{w_1}-\sqrt{n_1}d} \int_{w_1-(z+\sqrt{n_1}d)^2}^{h_1-(z+\sqrt{n_1}d)^2} F_{\chi_{p,\theta}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p-1,0}^2}(v) \phi(z) dv dz \\ &+ \int_{\sqrt{w_1}-\sqrt{n_1}d}^{\sqrt{h_1}-\sqrt{n_1}d} \int_0^{h_1-(z+\sqrt{n_1}d)^2} F_{\chi_{p,\theta}^2} \left(\frac{nh}{n_2} \right) f_{\chi_{p-1,0}^2}(v) \phi(z) dv dz \end{aligned}$$

where

$$\begin{aligned} \theta &= \frac{n_1}{n_2} \left[(z + \sqrt{n_1} d)^2 + v \right] + 2\sqrt{n_1} d Z + (2n_1 + n_2) d^2 \\ &= \frac{n_1}{n_2} \left[\left(z + \frac{n}{\sqrt{n_1}} d \right)^2 + v \right]. \end{aligned}$$

8 References

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Figure 1: Crankshaft to Piston Connecting Rod

Figure 2: Double Sampling T^2 Chart.

Tables 1-9

Table 1. $p = 3, n_0 = 3, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	120.900	22.451	27.047	17.942	28.872
		(10.48, 19.94, 6.12, 1, 19)	(8.01, 28.23, 8.20, 2, 24)	(13.41, 0.18)	(14.32, 7.09, 1, 20)
1.0	20.781	5.110	5.312	5.550	5.594
		(13.41, 14.86, 7.50, 2, 18)	(9.54, 19.61, 6.01, 2, 9)	(13.63, 0.22)	(14.32, 7.17, 2, 17)
1.5	5.161	2.008	2.133	2.95	2.595
		(16.08, 14.72, 6.20, 2, 10)	(13.86, 14.48, 6.78, 2, 13)	(14.11, 0.4)	(14.32, 6.25, 2, 12)
2.0	2.067	1.355	1.381	1.841	1.848
		(14.31, 14.99, 6.61, 2, 12)	(10.14, 15.80, 6.81, 2, 13)	(14.3, 0.72)	(14.32, 6.01, 2, 11)

Table 2. $p = 3, n_0 = 6, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	56.971	11.660	14.886	9.634	12.647
		(11.90, 16.40, 6.81, 4, 26)	(9.34, 19.51, 6.36, 4, 21)	(12.86, 0.12)	(14.32, 4.83, 1, 28)
1.0	6.337	2.304	2.518	3.249	2.892
		(16.06, 14.63, 7.61, 5, 19)	(15.93, 14.38, 6.69, 4, 25)	(14.01, 0.34)	(14.32, 7.17, 5, 20)
1.5	1.773	1.231	1.232	1.665	1.556
		(14.75, 14.77, 7.84, 5, 21)	(16.31, 14.35, 6.60, 5, 12)	(14.3, 0.72)	(14.32, 4.64, 5, 10)
2.0	1.097	1.017	1.041	1.107	1.169
		12.28, 18.96, 7, 5, 14	(9.67, 16.95, 6.55, 4, 23)	(14.32, 0.96)	(14.32, 7.58, 5, 23)

Table 3. $p = 3, n_0 = 10, n_1 + n_2 \leq 40$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	27.717	7.772 (12.63, 18.91, 3.40, 3, 21)	9.417 (13.02, 15.27, 4.31, 2, 35)	6.374 (13.63, 0.22)	7.96 (14.32, 5.35, 6, 33)
1.0	2.696	1.376 (18.00, 14.69, 5.70, 8, 16)	1.720 (14.24, 17.74, 3.61, 3, 23)	2.155 (14.26, 0.58)	1.928 (14.32, 6.47, 9, 20)
1.5	1.131	1.016 (16.03, 14.95, 6.95, 9, 14)	1.018 (10.28, 15.50, 6.44, 9, 11)	1.185 (14.29, 0.67)	1.162 (14.32, 7.58, 9, 27)
2.0	1.003	1.002 (14.56, 15.67, 5.15, 6, 25)	1.035 (15.68, 14.52, 4.08, 3, 28)	1.006 (14.3, 0.72)	1.007 (14.32, 7.17, 9, 24)

Table 4. $p = 5, n_0 = 3, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	155.895	24.346 (12.58, 26.63, 10.22, 1, 29)	29.580 (12.96, 20.98, 10.10, 1, 28)	18.95 (15.56, 0.06)	42.139 (18.386, 10.06, 1, 29)
1.0	30.382	5.507 (15.20, 23.70, 9.36, 2, 11)	6.355 (12.64, 21.79, 10.12, 2, 14)	6.278 (17.89, 0.27)	6.571 (18.386, 9.21, 1, 21)
1.5	7.262	2.739 (30.42, 18.41, 9.65, 2, 12)	2.735 (17.26, 18.57, 10.40, 2, 16)	3.512 (18.33, 0.59)	2.897 (18.386, 8.95, 2, 11)
2.0	2.619	1.349 (20.76, 18.88, 8.22, 2, 7)	1.359 (14.30, 20.30, 7.79, 2, 6)	2.11 (18.37, 0.74)	1.914 (18.386, 5.13, 1, 6)

Table 5. $p = 5, n_0 = 6, n_1 + n_2 \leq 30$

d	T^2	$DS\ T^2$	$DS\ Combined\ T^2$	$MEWMA$	$VSS\ T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	79.651	17.065 (15.62, 29.19, 7.43, 2, 21)	17.964 (12.27, 28.09, 9.94, 4, 26)	10.800 (17.2, 0.15)	18.259 (18.386, 7.51, 1, 28)
1.0	9.036	2.783 (17.09, 20.86, 7.12, 2, 19)	3.135 (13.95, 19.48, 10.08, 5, 14)	3.656 (18.24, 0.45)	3.466 (18.386, 10.49, 5, 21)
1.5	2.183	1.211 (18.20, 20.63, 8.93, 5, 9)	1.333 (19.38, 18.46, 8.88, 4, 18)	1.909 (18.34, 0.61)	1.712 (18.386, 8.25, 5, 12)
2.0	1.172	1.035 (16.81, 20.35, 10.27, 5, 15)	1.037 (20.44, 18.40, 10.05, 5, 14)	1.179 (18.38, 0.92)	1.212 (18.386, 7.29, 5, 10)

Table 6. $p = 5, n_0 = 10, n_1 + n_2 \leq 40$

d	T^2	$DS\ T^2$	$DS\ Combined\ T^2$	$MEWMA$	$VSS\ T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	40.268	7.723 (22.68, 32.72, 15.18, 1, 16)	10.796 (12.97, 23.92, 9.04, 7, 28)	7.28 (17.82, 0.25)	8.069 (18.386, 6.79, 1, 39)
1.0	3.556	1.683 (14.05, 22.32, 6.79, 6, 17)	2.104 (20.76, 18.50, 6.70, 3, 29)	2.47 (18.24, 0.45)	2.258 (18.386, 11.54, 9, 33)
1.5	1.225	1.044 (28.64, 18.43, 9.43, 8, 22)	1.049 (27.38, 18.39, 9.16, 8, 20)	1.235 (18.38, 0.82)	1.248 (18.386, 10.94, 9, 28)
2.0	1.008	1.001 (27.28, 18.42, 11.07, 9, 21)	1.070 (18.17, 20.08, 2.66, 1, 12)	1.011 (18.38, 0.83)	1.016 (18.386, 11.07, 9, 29)

Table 7. $p = 10, n_0 = 3, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	207.056	36.808	48.019	23.392	77.310
		(20.34, 36.81, 17.01, 1, 27)	(17.53, 50.44, 19.40, 2, 28)	(24.33, 0.07)	(27.112, 17.11, 1, 29)
1.0	51.84	7.636	8.024	7.717	8.452
		(21.96, 39.95, 15.62, 1, 18)	(20.71, 50.90, 15.62, 1, 18)	(16.43, 0.23)	(27.112, 16.46, 1, 24)
1.5	12.66	3.466	3.660	3.981	3.685
		(26.73, 33.13, 12.55, 2, 4)	(22.75, 28.53, 15.15, 2, 8)	(26.75, 0.32)	(27.112, 15.99, 2, 12)
2.0	4.07	1.588	1.737	2.633	2.355
		(30.75, 27.59, 14.09, 2, 6)	(24.87, 27.58, 15.13, 2, 8)	(26.84, 0.36)	(27.112, 15.61, 2, 11)

Table 8. $p = 10, n_0 = 6, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	121.071	25.056	28.964	19.939	30.310
		(23.37, 35.12, 14.30, 2, 25)	(22.96, 30.78, 13.73, 1, 27)	(26.03, 0.17)	(27.112, 14.14, 1, 31)
1.0	15.888	3.928	4.376	4.419	4.115
		(25.97, 9.54, 11.30, 3, 9)	(18.61, 33.25, 19.01, 5, 25)	(26.89, 0.39)	(27.112, 15.81, 4, 23)
1.5	3.263	1.406	1.827	2.320	2.019
		(25.59, 31.40, 16.08, 5, 10)	(23.05, 29.54, 11.30, 3, 9)	(27.01, 0.5)	(27.112, 13.81, 4, 15)
2.0	1.386	1.093	1.163	1.348	1.444
		(28.45, 27.69, 17.26, 5, 15)	(38.15, 27.11, 16.37, 4, 23)	(27.11, 0.89)	(27.112, 18.14, 5, 24)

Table 9. $p = 10, n_0 = 10, n_1 + n_2 \leq 40$

d	T^2	$DS\ T^2$	$DS\ Combined\ T^2$	$MEWMA$	$VSS\ T^2$
	$CL = \chi_{p,0.0025}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	67.005	13.161 (24.60, 30.15, 15.28, 6, 33)	22.035 (18.66, 34.87, 19.36, 9, 28)	8.892 (26.11, 0.18)	18.564 (27.112, 13.31, 4, 33)
1.0	5.808	1.859 (36.82, 27.29, 16.99, 9, 14)	2.422 (27.79, 27.25, 14.53, 6, 27)	3.250 (27.09, 0.7)	2.505 (27.112, 12.34, 5, 24)
1.5	1.488	1.068 (28.16, 46.84, 16.31, 9, 11)	1.108 (30.61, 27.16, 15.72, 8, 19)	1.537 (27.07, 0.62)	1.393 (27.112, 15.62, 9, 18)
2.0	1.029	1.003 (32.16, 27.45, 14.90, 8, 15)	1.003 (33.84, 27.11, 17.65, 9, 17)	1.03 (27.11, 0.89)	1.069 (27.112, 12.55, 8, 16)

Table 10. Summary Data for the example of application of the $DS\ T^2$ Chart.

Point number	$\overline{X}_{n_1=2}$	$\overline{X}_{n_2=18}$	T^2
1	$\begin{bmatrix} 19.72 & 6.95 & 4.022 \end{bmatrix}^T$	not measured	6.281
2	$\begin{bmatrix} 20.092 & 7.09 & 3.97 \end{bmatrix}^T$	not measured	4.043
3a	$\begin{bmatrix} 20.10 & 7.13 & 3.92 \end{bmatrix}^T$	to measure and combine	13.785
3b	$\begin{bmatrix} 20.10 & 7.13 & 3.92 \end{bmatrix}^T$	$\begin{bmatrix} 20.10 & 6.99 & 3.98 \end{bmatrix}^T$	12.04
4	$\begin{bmatrix} 19.94 & 6.99 & 4.09 \end{bmatrix}^T$	not measured	4.695
5	$\begin{bmatrix} 20.09 & 6.91 & 4.09 \end{bmatrix}^T$	not measured	15.359

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The following tables are presented for the reviewers convenience. Case $ARL(d = 0) = 1000$.

Tables I-IX

Table I. $p = 3, n_0 = 3, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	261.526	27.448	28.208	19.555	37.003
		(12.40, 19.10, 6.77, 1, 27)	(10.71, 21.66, 7.09, 1, 29)	(16.66, 0.09)	(16.266, 7.17, 1, 31)
1.0	37.028	5.660	5.725	6.354	5.648
		(14.92, 17.34, 6.11, 1, 19)	(12.85, 17.14, 7.12, 2, 16)	(15.69, 0.21)	(16.266, 6.01, 1, 19)
1.5	7.665	2.115	2.134	3.312	2.701
		(13.75, 19.41, 6.85, 2, 13)	(13.21, 16.90, 6.72, 2, 12)	(16.14, 0.42)	(16.266, 5.07, 2, 8)
2.0	2.614	1.354	1.468	2.106	2.092
		(22.31, 14.39, 6.60, 2, 12)	(12.77, 16.94, 7.43, 2, 17)	(16.22, 0.55)	(16.266, 7.17, 2, 17)

Table II. $p = 3, n_0 = 6, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	113.618	12.225	16.245	11.711	15.009
		(12.22, 17.36, 5.07, 2, 24)	(9.56, 18.72, 6.12, 4, 19)	(15.65, 0.2)	(16.266, 5.07, 1, 31)
1.0	9.698	2.386	2.464	3.642	2.904
		(17.74, 16.88, 7.28, 5, 16)	(12.64, 17.38, 5.87, 4, 17)	(16.12, 0.4)	(16.266, 4.64, 3, 18)
1.5	2.166	1.165	1.218	2.726	1.677
		(16.05, 18.14, 6.84, 5, 13)	(12.84, 16.75, 7.66, 5, 19)	(15.34, 0.15)	(16.266, 6.01, 5, 14)
2.0	1.159	1.022	1.013	1.187	1.230
		(18.89, 19.52, 3.40, 3, 9)	(12.11, 17.15, 5.43, 5, 7)	(16.26, 0.77)	(16.266, 6.47, 5, 16)

Table III. $p = 3, n_0 = 10, n_1 + n_2 \leq 40$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	51.052	8.772 (12.74, 24.02, 8.86, 9, 31)	13.683 (11.40, 17.09, 8.74, 9, 31)	7.509 (15.84, 0.25)	8.887 (16.266, 4.03, 2, 33)
1.0	3.594	1.379 (18.16, 17.18, 5.73, 7, 24)	1.417 (11.90, 18.07, 6.09, 7, 28)	2.428 (16.22, 0.55)	2.069 (16.266, 7.58, 9, 27)
1.5	1.21	1.073 (17.37, 17.73, 4.99, 5, 29)	1.021 (12.64, 17.37, 5.07, 7, 18)	1.236 (16.26, 0.77)	1.205 (1.230.013)
2.0	1.006	1.000 (15.78, 19.08, 17.30, 9, 16)	1.000 (23.56, 16.44, 6.33, 8, 21)	1.008 (16.27, 1)	1.013 (16.266, 6.47, 9, 20)

Table IV. $p = 5, n_0 = 3, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	347.843	35.299 (15.35, 24.63, 10.22, 1, 29)	50.517 (13.15, 27.66, 11.16, 2, 12)	22.64 (19.04, 0.1)	58.288 (20.515, 10.31, 1, 31)
1.0	56.536	6.360 (216.62, 29.14, 10.33, 2, 15)	6.816 (215.20, 25.36, 9.24, 1, 20)	7.174 (19.99, 0.22)	6.456 (20.515, 8.62, 1, 17)
1.5	11.305	2.442 (20.14, 21.68, 9.47, 2, 11)	2.958 (20.80, 20.60, 9.46, 2, 11)	3.79 (20.43, 0.46)	3.038 (20.515, 8.95, 2, 11)
2.0	3.465	1.365 (21.26, 25.50, 6.62, 2, 4)	1.907 (17.31, 20.83, 11.69, 2, 26)	2.388 (20.43, 0.46)	1.984 (20.515, 7.29, 2, 7)

Table V. $p = 5, n_0 = 6, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	165.007	24.719 (16.65, 26.93, 11.38, 5, 23)	30.237 (15.85, 23.21, 8.74, 3, 25)	13.17 (19.79, 0.18)	25.115 (20.515, 7.61, 1, 29)
1.0	14.491	3.126 (24.04, 20.76, 10.61, 5, 17)	3.788 (20.52, 20.50, 11.48, 5, 24)	4.174 (20.41, 0.43)	3.355 (20.515, 7.48, 3, 39)
1.5	2.787	1.249 (29.24, 20.73, 8.59, 5, 8)	1.400 (26.17, 20.44, 11.47, 5, 24)	2.154 (20.47, 0.54)	1.840 (20.515, 8.95, 5, 14)
2.0	1.27	1.028 (29.82, 20.88, 7.28, 4, 10)	1.058 (22.59, 20.54, 8.93, 4, 18)	1.466 (20.47, 0.53)	1.310 (20.515, 8.62, 5, 13)

Table VI. $p = 5, n_0 = 10, n_1 + n_2 \leq 40$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	77.361	11.609 (18.79, 26.20, 9.04, 7, 28)	13.743 (15.35, 24.89, 5.73, 2, 24)	8.914 (19.04, 0.1)	11.588 (20.515, 7.30, 3, 37)
1.0	4.966	1.515 (20.71, 21.76, 7.81, 7, 18)	2.502 (17.28, 20.73, 10.11, 9, 14)	2.72 (20.43, 0.46)	2.390 (20.515, 10.32, 9, 24)
1.5	1.348	1.033 (23.64, 21.39, 7.43, 7, 16)	1.068 (14.56, 30.96, 9.15, 7, 29)	1.36 (20.52, 0.97)	1.331 (20.515, 5.13, 8, 13)
2.0	1.015	1.004 (19.81, 23.94, 7.61, 6, 22)	1.004 (28.80, 20.50, 11.15, 9, 21)	1.076 (20.46, 0.52)	1.025 (20.515, 8.25, 9, 16)

Table VII. $p = 10, n_0 = 3, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	479.228	59.352	75.027	29.895	130.259
		(23.25, 40.90, 17.31, 1, 28)	(20.14, 39.84, 19.60, 2, 28)	(28.02, 0.1)	(29.588, 17.13, 1, 29)
1.0	102.813	8.057	9.373	12.937	8.552
		(23.31, 29.44, 15.77, 1, 19)	(20.46, 34.05, 16.07, 2, 12)	(29.47, 0.41)	(29.588, 15.98, 1, 21)
1.5	21.234	3.165	4.897	4.443	3.929
		(28.11, 39.66, 14.72, 2, 7)	(32.07, 29.69, 16.29, 1, 22)	(29.41, 0.36)	(29.588, 15.20, 2, 10)
2.0	5.807	1.647	1.939	2.978	2.321
		(37.14, 29.92, 14.12, 2, 6)	(48.74, 29.61, 16.57, 2, 12)	(29.47, 0.41)	(29.588, 17.42, 2, 9)

Table VIII. $p = 10, n_0 = 6, n_1 + n_2 \leq 30$

d	T^2	$DS T^2$	$DS Combined T^2$	$MEWMA$	$VSS T^2$
	$CL = \chi_{p,0.001}^2$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	263.884	42.995	45.557	16.055	52.456
		(26.12, 34.83, 16.35, 4, 22)	(23.57, 37.63, 14.93, 3, 25)	(28.53, 0.14)	(29.588, 13.87, 1, 29)
1.0	27.424	3.850	4.322	5.101	4.178
		(34.32, 30.57, 15.17, 5, 8)	(19.78, 29.59, 18.79, 5, 24)	(29.46, 0.14)	(29.588, 13.97, 2, 25)
1.5	4.487	1.708	1.843	2.706	2.178
		(42.21, 29.64, 18.68, 5, 23)	(38.33, 29.51, 15.09, 4, 16)	(29.58, 0.74)	(29.588, 12.53, 4, 12)
2.0	1.585	1.107	1.111	1.517	1.448
		(27.28, 32.24, 17.76, 5, 17)	(34.64, 29.59, 17.90, 5, 18)	(29.58, 0.72)	(29.588, 13.42, 5, 10)

Table IX. $p = 10, n_0 = 10, n_1 + n_2 \leq 40$

d	T^2	$DS\ T^2$	$DS\ Combined\ T^2$	$MEWMA$	$VSS\ T^2$
	$CL = \chi^2_{p,0.001}$	(h, h_1, w_1, n_1, n_2)	(h, h_1, w_1, n_1, n_2)	(h, r)	(CL, w, n_1, n_2)
0.5	136.763	15.611	17.758	10.750	24.521
		(20.30, 33.17, 7.51, 5, 27)	(20.40, 37.41, 15.87, 7, 29)	(27.86, 0.09)	(29.588, 13.12, 3, 35)
1.0	8.755	2.299	3.011	3.204	2.638
		(38.16, 29.65, 19.29, 9, 28)	(29.88, 30.07, 18.08, 9, 19)	(29.45, 0.39)	(29.588, 13.44, 5, 30)
1.5	1.735	1.129	1.179	1.614	1.532
		(49.46, 29.59, 18.18, 9, 23)	(42.22, 29.60, 17.72, 9, 17)	(29.58, 0.7)	(29.588, 11.32, 8, 14)
2.0	1.051	1.006	1.007	1.079	1.067
		(32.01, 50.91, 11.98, 6, 14)	(46.31, 29.59, 19.50, 9, 30)	(29.58, 0.71)	(29.588, 17.14, 9, 23)