Solving a stochastic first-order linear differential equation using the Karhunen-Loève expansion

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Motivation

Differential equations are useful tools to modeling numerous phenomena in many disciplines. From a practical standpoint, the application of differential equations requires setting their inputs (for example coefficients and initial conditions). These parameters are usually obtained by experiments, then measurament errors are involved. In addition, there are external sources which can affect the physical system to be modelled. These facts motivate us to consider the inputs parameters as a random variables (RVs) or stochastic processes (SPs) rather than deterministic constans or functions, respectively.

Notice that, the main problem dealing with a deterministic differential equation is to obtain the solution, however, in this case, the solution is a SP, say $X(t)$, and then it is also important to compute its main statistical functions, such as the mean, $\mu_X(t)$, and the variance, $\sigma_X(t)$. Nevertheless, a more convenient goal is the determination of its first probability density function (1-PDF), $f_1(x,t)$, because from it we have a full probabilist description of the solution SP in each time instant t. Furthermore, from the 1-PDF one can compute all the one-dimensional statistical moment of $X(t)$ and hence the mean and the variance can be easily obtained,

$$\mathbb{E}\left[(X(t))^k\right]=\int_{-\infty}^{\infty}x^k f_1(x,t)\,\mathrm{d}x,\quad k=0,1,2,\ldots,\quad \begin{cases}\mu_X(t)=\mathbb{E}\left[(X(t))^1\right]\\ \sigma_X(t)=\mathbb{E}\left[(X(t))^2\right]-(\mu_X(t))^2\end{cases}\tag{1}$$

In this work, we will solve an initial value problem (IVP) based on a first-order linear equation where the linear coefficient, $a(t,\omega)$, is a SP and the initial condition, $X_0(\omega)$ is a RV. Then, our purpose is to obtain the first probability density function (1-PDF), $f_1(x,t)$, of the solution SP, $X(t)$, of the following problem

$$\left.\begin{array}{rcl}X'(t,\omega)&=&a(t,\omega)X(t,\omega),\\ X(t_0,\omega)&=&X_0(\omega).\end{array}\right\}\ t\in\mathcal{T}\subset\mathbb{R}^+.\tag{2}$$

In order to obtain the 1-PDF of the solution SP $X(t)$ of the IVP (2) both the Karhunen-Loève Expansion (KLE) and the Random Variable Transformation (RVT) technique will be used. Combinig both techniques the 1-PDF of the truncated solution SP, $f_1^N(x,t)$, will be obtained. Then, we impose a mild conditions with which the convergence in distribution is assured. Both results are stated as follows

Theorem: RVT technique. [1, pp. 24–25]

Let $\mathbf{U}=(U_1,\ldots,U_n)^\top$ and $\mathbf{V}=(V_1,\ldots,V_n)^\top$ be two n-dimensional absolutely continuous random vectors. Let $\mathbf{g}:\mathbb{R}^n\to\mathbb{R}^n$ be a one-to-one deterministic transformation of $\mathbf{U}$ into $\mathbf{V}$, i.e., $\mathbf{V}=\mathbf{g}(\mathbf{U})$. Assume that $\mathbf{g}$ is continuous in $\mathbf{U}$ and has continuous partial derivatives with respect to $\mathbf{U}$. Then, if $f_{\mathbf{U}}(\mathbf{u})$ denotes the joint PDF of vector $\mathbf{U}$, and $\mathbf{h}=\mathbf{g}^{-1}=(h_1(v_1,\ldots,v_n),\ldots,h_n(v_1,\ldots,v_n))^\top$ represents the inverse mapping of $\mathbf{g}=(g_1(u_1,\ldots,u_n),\ldots,g_n(u_1,\ldots,u_n))^\top$, the joint PDF of vector $\mathbf{V}$ is given by

$$f_{\mathbf{V}}(\mathbf{v})=f_{\mathbf{U}}(\mathbf{h}(\mathbf{v}))\,|J|,$$

where $|J|$ is the absolute value of the Jacobian.

L^2 convergence of KLE [2, pp. 202]

Consider a SP $\{a(t)\in L^2(\Omega,L^2(\mathcal{T}))$, with mean $\mu_a(t)$. Then,

$$a(t,\omega)=\mu_a(t)+\sum_{j=1}^{\infty}\sqrt{\nu_j}\phi_j(t)\xi_j(\omega),\tag{3}$$

being $\{\nu_j,\phi_j\}$ denote, respectively, the eigenvalues and eigenfunctions of the covariance function, $C(s,t)$. Random variables $\xi_j(\omega)$ have mean zero, unit variance and are pairwise uncorrelated.

Theoretical Results

To apply the RVT technique, from a computational point of view, we consider the N-truncation of (3). Then, the 1-PDF, $f_1^N(x,t)$, of the truncated solution SP, $X_N(t)$,

$$f_1^N(x,t)=\int_{\mathcal{D}(\xi)}f_{X_0,\xi_1,\ldots,\xi_N}\left(x\ \mathrm{e}^{-\int_{t_0}^{t}a_N(s,\omega)\mathrm{d}s},\xi_1,\ldots,\xi_N\right)\mathrm{e}^{-\int_{t_0}^{t}a_N(s,\omega)\mathrm{d}s}\,\mathrm{d}\xi_N\cdots\mathrm{d}\xi_1.\tag{4}$$

And finally, we establish conditions for the uniform convergence.

- $\mathcal{D}(\xi)=\mathcal{D}(\xi_1,\ldots,\xi_N)$ is bounded.
- $\forall N,\quad f_{X_0,\xi_1,\ldots,\xi_N}(x_0,\xi_1,\ldots,\xi_N)$ is uniformly bounded.
- $\mathcal{T}$ is closed and bounded.
- The covariance function is continuous in $\mathcal{T}\times\mathcal{T}$.
- $g(x_0)=f_{X_0,\xi_1,\ldots,\xi_N}(x_0,\xi_1,\ldots,\xi_N)$ is continuous in x_0 $\forall N$ fixed.

. Then,

$$\lim_{N\to\infty}f_1^N(x,t)=f_1(x,t),\quad \forall(x,t)\in\mathbb{R}\times\mathcal{T}\text{ fixed.}$$

Graphical Example

We consider that the SP $a(t,\omega)$ is the Brownian motion, $t_0=0$ and $T=2$. Then, it is known that the covariance function is $C(s,t)=\min(s,t)$, $(s,t)\in\mathcal{T}\times\mathcal{T}$, $\mu_a(t)=0$ and $\sigma_a(t)=1$, $\forall t\in\mathcal{T}=[0,T]$. Notice that, this is a test example, because of the 1-PDF, $f_1(x,t)$, of the exact solution SP, $X(t)$, can be computed

$$f_1(x,t)=\int_{-\infty}^{\infty}f_{X_0,W}(x\,\mathrm{e}^{-w},w)\,\mathrm{e}^{-w}\,\mathrm{d}w.\tag{5}$$

In Figure 1 the graphical representation of the 1-PDF of the exact and truncations $N=1$ and $N=2$ are showed taking the following distributions for the input parameters

- $\xi_j\sim\mathrm{N}(0,1)$, $j=1,2$, truncated in the interval $[-10,10]$..
- $X_0\sim\mathrm{Un}[0,1]$.
- X_0, ξ_1 and ξ_2 independent RVs.

We observe that the 1-PDF of the first truncation is closed to the 1-PDF of the exact solution. For sake of clarity in Figure 2 the 1-PDFs for this truncation in different times are computed and in Table 3 the error,

$$e_N=\int_{-\infty}^{\infty}\left|f_1(x,t)-f_1^N(x,t)\right|\mathrm{d}x,\quad t\in\{0.1,1,2\},\ N\in\{1,2\},\tag{6}$$

is calculated for this times levels.

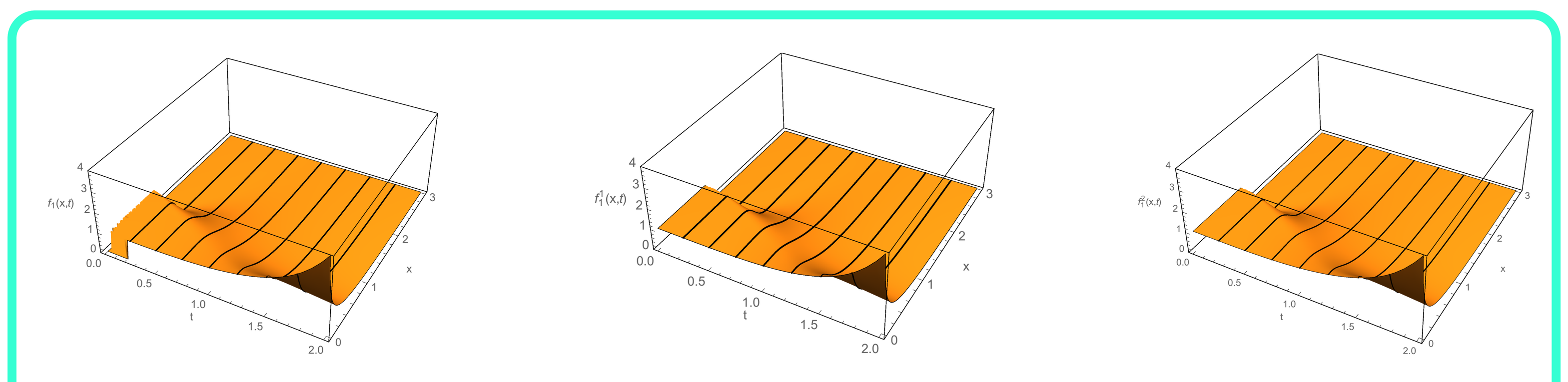

Fig. 1: Left: 1-PDF of the exact solution SP given by (5). Center, right: 1-PDF of the first and second truncation, respectively given by (4).

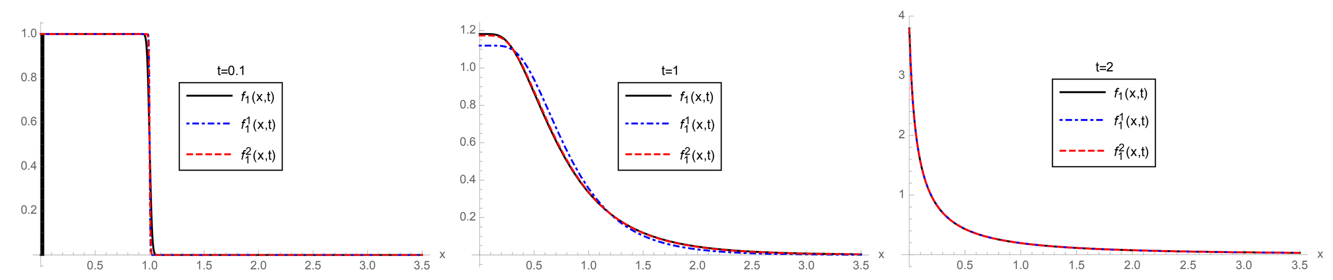

Fig. 2: 1-PDF of the exact solution SP and the truncation $N=1,2$, for differents values of t. Left: $t=0.1$. Center: $t=1$. Right: $t=2$.

e_N	$N=1$	$N=2$
t=0.1	0.019319	0.016788
$t=1$	0.077919	0.008663
$t=2$	0.005310	0.000832

Fig. 3: Error measure e_N defined by (6) for different time instants, $t\in\{0.1,1,2\}$, and truncation orders, $N\in\{1,2\}$.

Finally, in Figure 5 the mean and the standard deviation for different truncation are plotted. We can observe the convergence of both, although the standard deviation is less fast. As before the error is calculated in Table 4 using the expression (6) but in this case taking as a functions the mean and the standard deviation.

e_N	$N=1$	$N=2$	$N=3$	$N=4$
Mean	0.055567	0.005541	0.002425	0.000871
Standard deviation	0.383975	0.169942	0.159339	0.151808

Fig. 4: Error measure e_N defined by (6) to the mean for truncations $N=1,2$ and the standard deviation for truncations $N=1,2,3,4$.

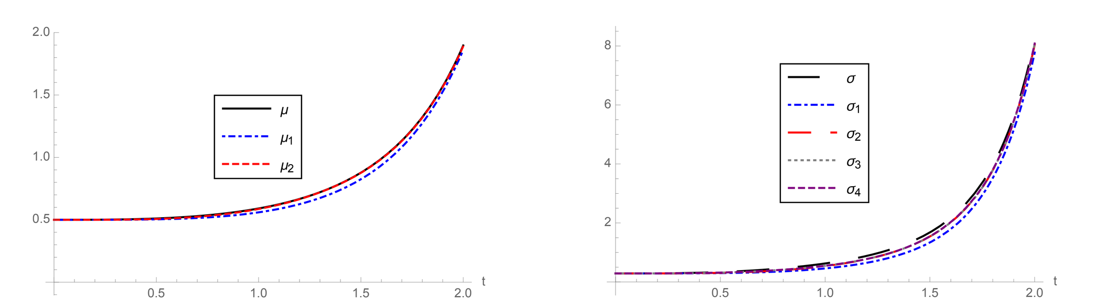

Fig. 5: Left: Mean of the solution and truncations $N=1$ and $N=2$. Right: Standard deviation of the solution and truncation $N=1,2,3,4$

References

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