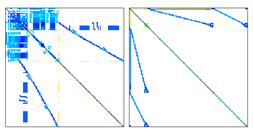
Preconditioners for Nonsymmetric Linear Systems with Low-Rank Skew-Symmetric Part



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MOTIVATION AND QUESTION CONSIDERED

PROBLEM

We study the iterative solution of linear systems

$$Ax = b, \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular, large and sparse. Let $A = H + K$, where H and K are its symmetric and skew-symmetric parts, respectively. Assume K can be approximated by a low-rank matrix and that $K = FCF^T + E$ where $F \in \mathbb{R}^{n\times s}$ and $C \in \mathbb{R}^{s\times s}$ (skew-symmetric matrix) have full rank with $s \ll n$, $\| E \| \ll 1$.

ACTUAL STATE

Different strategies have been proposed to solve (1), when $E = 0$:

- Progressive GMRES (PGMRES) [1]: uses short recurrence formulas and suffers from instabilities due to the loss of orthogonality.
- Schur complement method (SCM) [3]: applies the MINRES method $s+1$ times. It can be used as a preconditioner for GMRES, but can be costly.

WHAT WE PROPOSE

Compute a preconditioner for the matrix $\bar{A} = H + FCF^T$ that approximates A, following the strategy presented in [2]. The preconditioner is obtained from an approximate block LDU factorization of the augmented matrix

$$\begin{pmatrix} H & F \\ F^T & -C^{-1} \end{pmatrix}. \qquad (2)$$

It is used as preconditioner for the (restarted) GMRES [6] and BICGSTAB [7] methods.

OUR UPDATED PRECONDITIONER METHOD (UP) AND SOME NUMERICAL RESULTS

Preconditioner Computation

The block LDU factorization of the matrix in (2) is:

$$\begin{pmatrix} H & F \\ F^T & -C^{-1} \end{pmatrix} = \begin{pmatrix} L_H & 0 \\ F^T U_H^{-1} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} U_H & L_H^{-1}F \\ 0 & I \end{pmatrix}$$

where $R = -(C^{-1} + F^T U_H^{-1} L_H^{-1} F)$.

1. Compute $H \approx L_H U_H$.
2. Compute $T_1 = F^T U_H^{-1}$ and $T_2 = L_H^{-1} F$.
3. Compute $R = -(C^{-1} + T_1 T_2)$.
4. Compute $R \approx L_R U_R$.

Preconditioner Application

Obtain the preconditioned vector s from:

$$\begin{pmatrix} L_H & 0 \\ F^T U_H^{-1} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} U_H & L_H^{-1}F \\ 0 & I \end{pmatrix} \begin{pmatrix} s \\ s_1 \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

1. Solve $L_H r_1 = r$.
2. Solve $(L_R U_R) r_2 = -T_1 r_1$.
3. Solve $U_H s = r_1 - T_2 r_2$.

The computation and application of the preconditioner is inexpensive provided that $s \ll n$. The preconditioner can be viewed as a low-rank update (see [2]) of the incomplete factorization computed for H. It will be referenced as updated preconditioned method (UP).

Numerical Results

EXAMPLE 1. See [3]. Consider

$$A = \begin{bmatrix} \Lambda_- & & \\ & \Lambda_+ & \\ & & Z \end{bmatrix}$$

where $\Lambda_- = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $\Lambda_+ = \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_{n-s})$ and $Z = \mathrm{tridiag}(-\gamma, 1, \gamma) \in \mathbb{R}^{s\times s}$, $p \ll n$, $2 \le s \ll n$ and $\gamma > 0$. The eigenvalues are:

- $\lambda_1, \ldots, \lambda_p$ uniformly spaced in the negative real interval $[-\beta, -\alpha]$, $0 < \alpha < \beta$;
- $\lambda_{p+1}, \ldots, \lambda_{n-s}$ uniformly spaced in the positive real interval $[\alpha, \beta]$;
- and the other s eigenvalues from the skew-symmetric matrix Z.

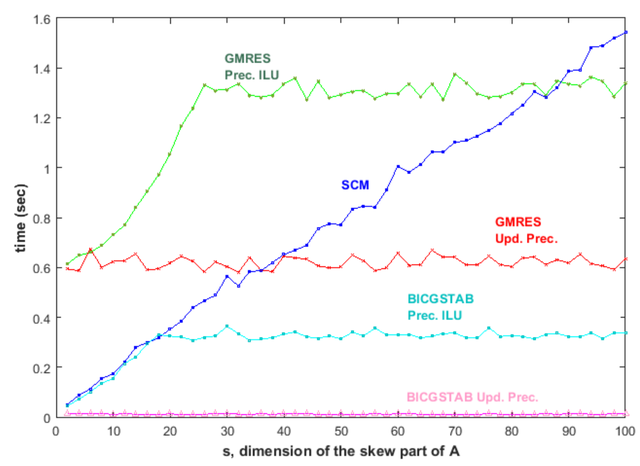

Figure: *Time comparison to solve the system $Ax = b$ with $b = 1/\sqrt{n}$, $n = 10^5$, $\alpha = 1/8$, $\beta = 1$, $\gamma = 1$. Preconditioner density is equal to 2 in all cases.*

EXAMPLE 2. See [1]. Bratu 2D problem consists on solving the non linear boundary problem

$$-\Delta u - \lambda \exp(u) = 0 \text{ in } \Omega, \text{ with } u = 0 \text{ on } \partial\Omega \quad (3)$$

depending on the parameter λ, Δ is the Laplacian, Ω the unit square and $\partial\Omega$ its boundary. Discretized with five-point stencil finite difference, in a grid of 500×500 points. The coefficient matrix has order $n = 2.5 \times 10^5$ with skew symmetric part of rank 2.

Method	Time (s)	Iter
GMRES[100] IC	45,1028	123
GMRES[100] UP	46,2829	131
BICGSTAB	26.6754	827
BICGSTAB IC	13,1653	194
BICGSTAB UP	11,2569	156
SCM	38,2014	255

Table: *Incomplete Cholesky factorization of H with dropping of 10^{-2}. Preconditioner density is equal to 0.7131.*

EXAMPLE 3. Define:

$$A = \begin{bmatrix} \Psi & & \\ & \Gamma & \\ & & \Omega \end{bmatrix}, \quad FCF^T = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \Omega \end{bmatrix}, \quad E = \begin{bmatrix} 0 & & \\ & \Gamma & \\ & & 0 \end{bmatrix}$$

where Ψ is of size $n/2$ from the discretization of the 2D Poisson operator, $\Gamma = \mathrm{tridiag}(-\gamma, -4, \gamma)$ and $\Omega = \mathrm{tridiag}(\omega, -4, \omega)$ are tridiagonal matrices of dimension $n/2 - s$ and $s \ll n$, respectively. We consider $n = 250000$, $\gamma = 0.01$, $\omega = 10$ and s taking different even values (particularly from 10 to 40) representing the rank of FCF^T that approximates K. In this case $||E|| = 0.02$.

Eigenvalue distribution of A:

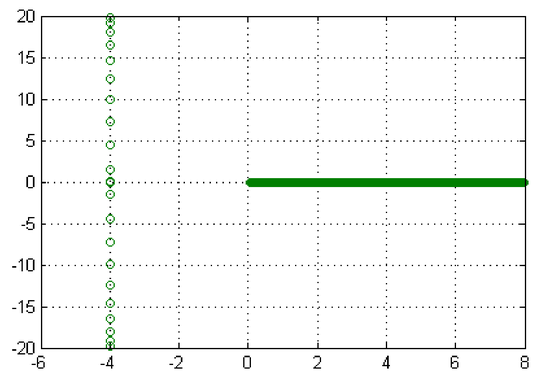

Figure: *Eigenvalues for A with n=2500 and s=20.*

Results: We use an incomplete LU of the symmetric part H with drop tolerance 10^{-2}, 1000 as maximum number of iterations to reach convergence with residual 10^{-8}. The right hand side b is a random vector.

Iterations				
s	10	20	30	40
GMRES[90] ILU H	228	233	274	398
GMRES[90] UP	99	99	99	99
GMRES[90] SCM P.	206	206	206	206
BISGSTAB ILU H	259.5	663.5	993	†
BICGSTAB UP	114	125	113	125

Time (sec.)				
s	10	20	30	40
GMRES[90] ILU H	75.3	79.8	101.7	140.0
GMRES[90] UP	36.0	35.8	36.3	36.2
GMRES[90] SCM P.	73.6	74.2	76.3	77.6
BISGSTAB ILU H	14.6	37.3	56.4	†
BICGSTAB UP	6.7	7.3	6.7	7.4

The results were obtained with MATLAB.

Future Job

1. We already have some spectral properties of our preconditioner.
2. Find out applications satisfying our assumptions and test our method.
3. Implement Balanced Incomplete Factorization, see [4, 5].

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