Weighted Banach spaces of harmonic functions

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Summary

The Ph.D. thesis "Weighted Banach Spaces of harmonic functions" presented here, treats several topics of functional analysis such as weights, composition operators, Fréchet and Gâteaux differentiability of the norm and isomorphism classes.

Chapter 1. Harmonic associated weights

Let U be an open and connected set of $\mathbb{R}^d$. A weight on U is a continuous function $v : U \to]0, \infty[$. For a weight v the *weighted Banach spaces of harmonic functions* with weight v are defined by:

$h_v(U) := \{f \in h(U) : \|f\|_v := \sup_{z\in U} v(z)|f(z)| < \infty\}$,
$h_{v_0}(U) := \{f \in h(U) : vf \text{ vanishes at infinity on } U\}$.

The harmonic associated weight is defined as

$$\widetilde{v}_h(z) := \frac{1}{\sup\{|f(z)| : \|f\|_v \leq 1\}}.$$

1. Aims

- Explain its properties.
- Compare the harmonic and holomorphic associated weights.
- Find differences and conditions under which they are the same or equivalent.
- Study the behavior with changes in the norm of $\mathbb{R}^d$ and in the dimension.

Chapter 2. Composition operators

If G_1 and G_2 are open and connected subsets in $\mathbb{C}^N$ and $\mathbb{C}^M$ and $\varphi : G_2 \to G_1$ is a holomorphic function, then we can consider the composition operator

$$C_\varphi : ph(G_1) \to ph(G_2),$$
$$C_\varphi(f) := f \circ \varphi,$$

where $ph(G_1)$ and $ph(G_2)$ are spaces of pluriharmonic functions.

2. Aims

Characterize

- the continuity, the compactness and the essential norm of C_φ

between weighted spaces of pluriharmonic functions $ph_{v_1}(G_1)$ and $ph_{v_2}(G_2)$.

Chapter 3. Smoothness of the norm

Let X be a real Banach space and let X^* denote its topological dual.

Šmulyan criterion

1. The norm of X is Gâteaux differentiable at $x \in X$ if and only if there exists x^* in the unit ball of X^* weak* exposed by x.
2. The norm is Fréchet differentiable at x if and only if x^* is weak* strongly exposed in the unit ball of X^* by x.

3. Aims

- Find a good approach of differentiability.
- Get applications for spaces of harmonic and continuous functions.

Chapter 4. Isomorphism classes

A Banach space X is said to be rotund if $\left\|\frac{x+y}{2}\right\| < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y$, $x, y \in X$.

4. Aims

- Show that there is an isomorphism between $h_{v_0}(U)$ and a closed subspace of c_0.
- Study under which conditions there is an isometry and whether these properties have some connection with the rotundity of the space.
- Look for conditions on the weight to know the rotundity of the space $h_{v_0}(U)$.

Learning

- About weighted Banach spaces of holomorphic functions and composition operators: Bierstedt, Bonet, Collado, Contreras, Domański, Frerick, Galbis, García, Hernández-Díaz, Jordá, Jornet, Laitilla, Lindström, Maestre, Montes, Rueda, Sevilla-Peris, Taskinen, Tylli, Wolf and others. (Since a work of Williams in 1967 about growth condition).
- About spaces of harmonic functions in connection with the growth of the harmonic conjugate of a function. Results of duality for weighted spaces of harmonic functions on the open unit disc: Shields and Williams (1978, 1982).
- About smoothness. Stolz's book (1893), Gâteaux differentiability (1913, 1919, 1922), Fréchet differentiability (1911, 1925), Šmulyan (1939, 1940), Cox and Nadler (1971), Heinrich (1975), Hennefeld (1979), Leonard and Taylor (1983), Boyd and Rueda (2006).
- About isomorphism classes and geometry: Lusky (1992, 1995, 2006), Kalton and Werner (1995), Bonet and Wolf (2003), Boyd and Rueda (2003, 2005, 2006).

Some results

Theorem

Let B be the unit ball of $(\mathbb{C}^N, |\cdot|)$ and let $\varphi : G \to B$ a holomorphic function on an open and connected set G in $\mathbb{C}^M$. Let $g : [0, 1[\to \mathbb{R}^+$ be a continuous function with $g(1^-) = 0$ and let $v(z) = g(|z|)$ be weight on B such that $\widetilde{v}_H = \widetilde{v}_{ph}$. Let w a weight on G which vanishes at ∞. Suppose that the operator $C_\varphi : ph_v(B) \to ph_w(G)$ is continuous. Then

$$\|C_\varphi\|_e = \limsup_{|\varphi(z)|\to 1} \frac{w(z)}{\widetilde{v}(\varphi(z))}$$

where $\widetilde{v}$ denotes the common associated weight for the spaces of holomorphic and pluriharmonic functions.

Counterexample

We define the function $g : \partial\mathbb{D} \to \mathbb{R}$ by: $g(e^{it}) = \left|1 - e^{it}\right|^2$, $t \in [-\pi, \pi]$, $g \in L^1(\partial\mathbb{D})$, is continuous, $g(1) = 0$ and $g \geq 0$. Let $w : \mathbb{D} \to \mathbb{R}$ be the Kernel Poisson of the function g, that is, $w(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) g(e^{it})dt$. One can show $\sup_{r\in]0,1[} \frac{\widetilde{v}_H(r)}{\widetilde{v}_h(r)} = \infty$.

Theorem

Let U be an open set in $\mathbb{R}^d$ and let v be a weight on U such that $h_{v_0}(U)$ contains the polynomials of degree smaller or equal than 1. Let X be a real Banach space. The norm $\|\cdot\|_v$ is Gâteaux differentiable at $f \in h_{v_0}(U, X)$ if and only if there exists $z_0 \in P_v$ peaking f, with the norm of X being Gâteaux differentiable at $v(z_0)f(z_0)$. In this case $\|\cdot\|_v$ is also Gâteaux differentiable at f in $h_v^c(U, X)$ and the norm $\|\cdot\|_v$ is Fréchet differentiable at f considered as a function in $h_v^c(U, X)$ if and only $\|\cdot\|$ is Fréchet differentiable at $v(z_0)f(z_0)$.

Theorem

Let U be a connected open subset of $\mathbb{R}^d$ and let v be a weight on U. Then, the space $h_{v_0}(U)$ is isomorphic to a closed subspace of c_0. More precisely, for each $\varepsilon > 0$ there exists a continuous linear injective map with closed range $T : h_{v_0}(U) \to c_0$ such that

$$(1 - \varepsilon)\|f\|_v \leq \|T(f)\| \leq \|f\|_v$$

for each $f \in h_{v_0}(U)$.

Proposition

Let $g : [0, 1] \to [0, \infty[$ be a continuous decreasing function such that $g(1) = 0$ and $\log(1/g)$ is strictly convex in $[0, 1[$.

(a) *Let $U \subseteq \mathbb{C}^d$ be the unit ball for a norm $|\cdot|$ and let $v : U \to]0, \infty[$, $x \mapsto g(|x|)$. Then $H_{v_0}(U)$ is not rotund (and then neither $h_{v_0}(U)$ is).*
(b) *Let $U \subseteq \mathbb{R}^d$ be the Euclidean unit ball and let $v(x) = g(\|x\|)$ for $x \in U$. Then $h_{v_0}(U)$ is not isometric to any subspace of c_0.*

References

- E. Jordá and A. M. Zarco, "Isomorphisms on weighed Banach spaces of harmonic and holomorphic and harmonic functions", *Journal of Function Spaces and Applications*, 2013.
- E. Jordá and A. M. Zarco, "Weighted Banach spaces of harmonic functions", *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 108 (2): 405-418, 2014.
- E. Jordá and A. M. Zarco, "Smoothness in some Banach spaces of functions". Preprint, 2014.