Rational Lanczos reduction of groundwater flow models to perform efficient simulations of surface-ground water interaction in conjunctive use systems

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INTRODUCTION: The Rational Lanczos method (RLANRM) [1] has been used in the last twenty years to reduce groundwater flow models of several kinds of aquifers. Generally, those aquifers have been spatially discretized via finite elements [2], but in this paper we present an adaptation of RLANRM to simulate surface-ground water relationships via finite differences (FD) in conjunctive use systems. First, the mathematical framework to solve the partial differential equation (PDE) of groundwater flow via RLANRM is presented. Later, the algorithm for the reduced simulation is discussed. Finally, relevant results of applying RLANRM on rectangular aquifers are shown.

GOAL: In this poster we propose: (i) a conceptual framework to reduce groundwater flow models via the rational Lanczos reduction method (RLANRM), (ii) an efficient evaluation of surface-ground water interaction, (*ii*) efficient algorithms for the generation of the Krylov's subspaces and (*iii*) physically based criteria to stop the Lanczos' vectors generation.

MATHEMATICAL FRAMEWORK OF RLANRM IN CONJUCTIVE USE SYSTEMS MODELATION

Let's consider a linear, time invariant aquifer, whose spatial domain has been discretized in *n* active FD nodes to solve the groundwater flow PDE. The aquifer's hydraulic parameters and boundary conditions don't change in time and the superposition principle applies. Let $h(t) \in R^n$ be the vector of piezometric heads in each active FD node of the model [L]. It can be written that h(*t*) = u + w(*t*), where $u \in R^n$ is the steady state solution subjected to the boundary conditions imposed to the original groundwater model and $w(t) \in R^n$ is the transient solution for the following dynamical system:

$$\mathbf{A}\mathbf{w}(t) + \boldsymbol{\psi}\mathbf{r}(t) = \mathbf{S}\, d\mathbf{w}(t)/dt \qquad (1)$$

where $\mathbf{A} \in R^{n\,x\,n}$ is the conductances matrix [L²/T], $S \in R^{n\,x\,n}$ is the matrix of storages [L²], $\psi \in R^{n\,x\,n_a}$ is the matrix of time invariant elemental excitations acting over the aquifer, n_a is the number of external actions (EA) and $\mathrm{r}(t) \in R^{n_a}$ is the vector of intensity for the AE [L³/T]. The boundary conditions imposed to (1) are zero and its initial condition is w(0)=h(0)-u [3]. Assuming that the intensities of the EA are zero and w(*t*)=VΦ(*t*), where $V \in R^{n\,x\,n}$ is an orthogonal projection matrix and $\Phi(t) \in R^n$ is a vector of states, substituting w(*t*) in (1) and applying the variable's separation technique, two problems are obtained [3]: (*i*) a generalized eigenvalue problem and (*ii*) a time dependent first order differential equation. Applying the inverse transformation combined with a spectral shift [2], [4], the previously mentioned generalized eigenvalue problem can be expressed as:

$$(\mathbf{A} - \sigma\mathbf{S})^{-1}\mathbf{S}\mathbf{V} = \boldsymbol{\Omega}\mathbf{V} \qquad (2)$$

$\Omega \in R^{n\,x\,n}$ is the diagonal matrix of shifted eigenvalues. Executing *m* steps of the rational Lanczos iteration [4], the equation (2) can be expressed as follows [2]:

$$\mathbf{X}^*\mathbf{S}(\mathbf{A} - \sigma\mathbf{S})^{-1}\mathbf{S}\mathbf{X} = \boldsymbol{\Pi};\ \mathbf{X}^*\mathbf{S}\mathbf{X} = \mathbf{I} \qquad (3)$$

$\mathrm{X} \in R^{n\,x\,m}$ is the matrix of Lanczos vectors, $\Pi \in R^{m\,x\,m}$ is the tri-diagonal matrix of Lanczos. The *m* eigenvalues of Π are good approximations of *m* eigenvalues of (2).

The participation factor for the i^{th} Lanczos vector is defined as the proportion which contributes to the aggregate volume of the EA, as follows:

$$p_{i,j} = \left(\sum_{k=1}^{n} s_{k,k}\, x_{k,i}\right) \mathbf{x}_i^{*} \psi_j \qquad (4)$$

with *i*=1,…,*m* and *j*=1,…,n_a, *x* and *s* are the elements of X and S, x_i, ψ_j are the i^{th} and j^{th} column vectors of X and Ψ, respectively. The accumulated participation factors, pa_j with *j*=1,…,n_a, are calculated as criteria to stop the Lanczos iteration, because they have unitary upper limits. Thus, when pa_j is close enough to one for all EA acting over the aquifer, the generation of the Krylov's reduction subspace is stopped. Replacing (3) in (2), defining that $\mathrm{A}_\sigma = (\mathbf{A} - \sigma S)$ and using the S-orthonormality of Lanczos vectors, the equation for the aquifer's states ($\Phi(t) \in R^m$) is obtained [2]:

$$\boldsymbol{\Pi} d\Phi(t)/dt - \mathbf{G}\mathbf{r}(t) = (\sigma\boldsymbol{\Pi} - \mathbf{I})\Phi(t) \qquad (5)$$

where $\mathbf{G} = \mathbf{X}^*\mathbf{S}\mathbf{A}_\sigma\boldsymbol{\psi} \in \mathrm{R}^{m\,x\,n_a}$ and w(*t*)=XΦ(*t*) is the approximated solution for transient piezometric heads. The Lanczos states are also approximated using an implicit FD for the time dependent derivative. Therefore, assuming that transient simulations are performed using time intervals of equal duration Δ*t* and the EA's intensities are constant during those intervals, the following expression is obtained:

$$\left(\frac{\boldsymbol{\Pi}}{\Delta t} + \mathbf{I} + \sigma\boldsymbol{\Pi}\right)\Phi^{t+1} = \frac{\boldsymbol{\Pi}}{\Delta t}\Phi^{t} + \mathbf{G}\mathbf{r}(\tau) \qquad (6)$$

which is a tri-diagonal system of linear equations where the vector of Lanczos states at time *t*+1 is unknown. Equation (6) is solved using bi-conjugate gradient solvers. Finally, to calculate the integrated volumes of surface-ground water interaction, the time integration of Lanczos' states is executed numerically via Simpson's rule.

MODELS CONFIGURATIONS

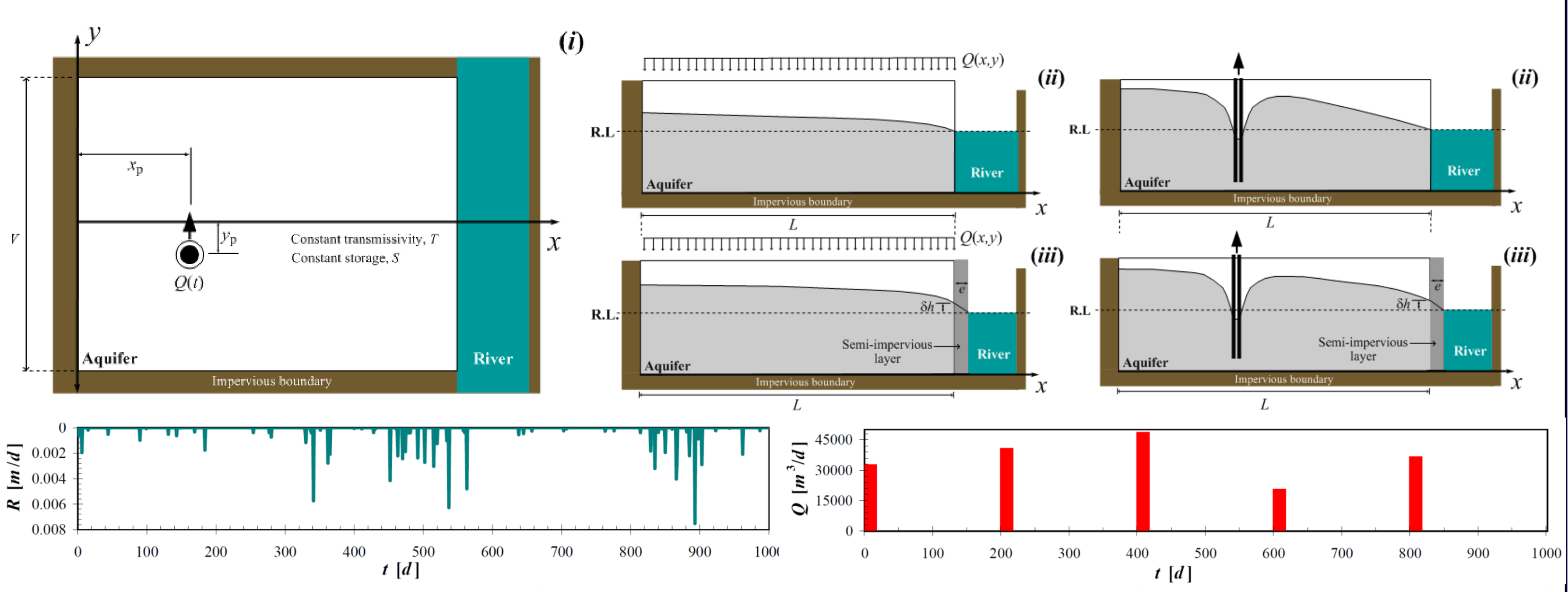

Fig. 1 From left to right: Aquifer's configuration for uniformly distributed recharge, punctual pumping. Down: the intensities of the external actions (EA).

KRYLOV'S SUBSPACES GENERATION

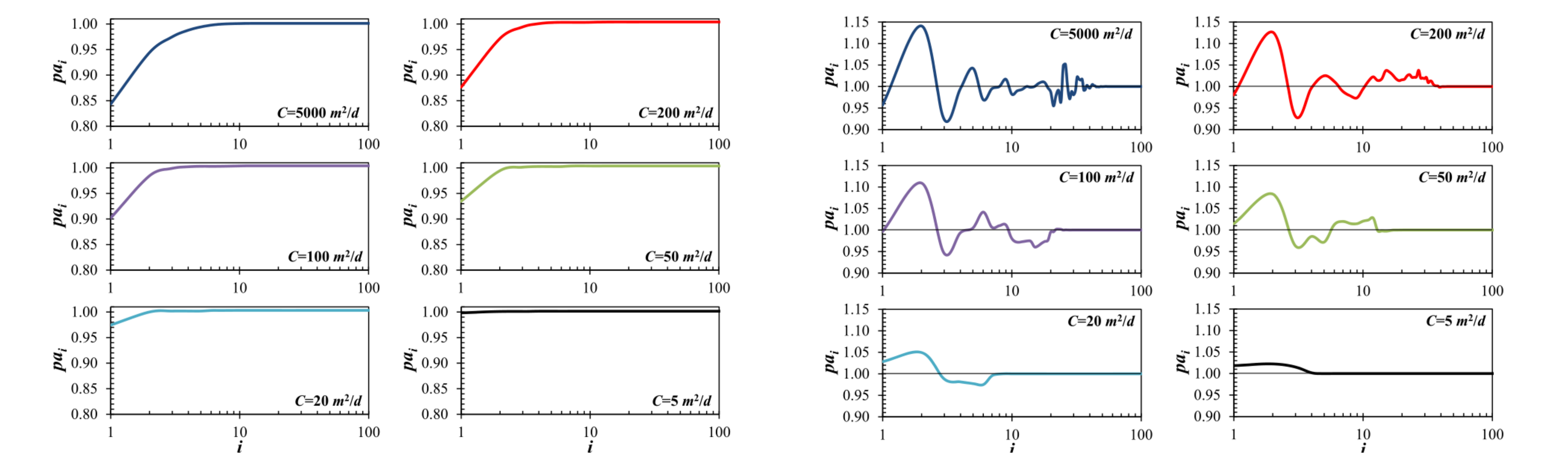

Fig. 2 From left to right: Accumulated participation factors for the uniform recharge and for the punctual pumping as a function of the riverbed's conductance (*C*.)..

RESULTS OF GROUNDWATER SIMULATIONS

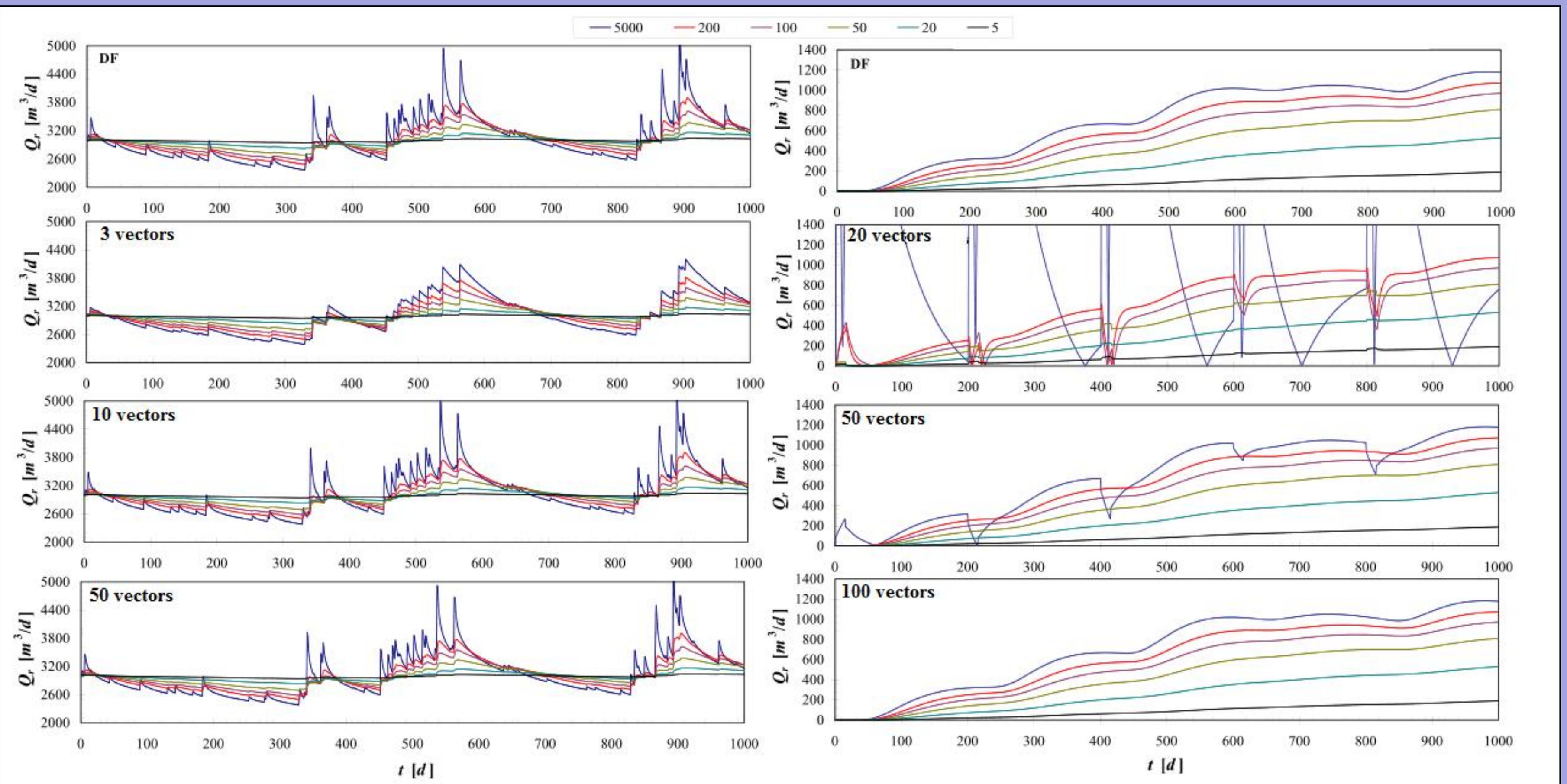

Fig. 3 Simulated groundwater hydrographs for different sizes of the Krylov's reduction subspace as a function of the riverbed's conductance for recharge (left) and pumping (right).

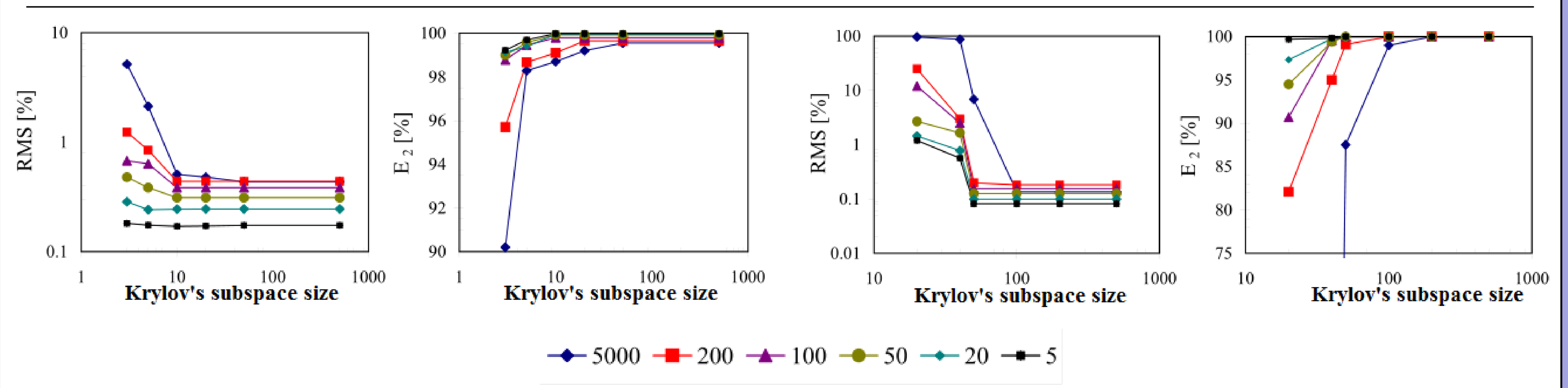

Fig. 4 Performance indices estimated for the surface-ground water relationships simulated with RMLANRM in homogeneous rectangular aquifer for recharge (left) and pumping (right).

CONCLUSIONS: When the *C* is low, the RLANRM's performance improves because fewer Lanczos vectors are needed to obtain more accurate simulations. On the contrary, when river-aquifer connection is almost perfect, the reduced model's accuracy decreases and the cost of building the Lanczos' vectors augments. Numerical experiments have shown that the classical eigenvalue method (EMV) [3] is more efficient than RLANRM to perform the simulation of the transient river-aquifer interactions because it possesses a simple explicit state equation. This disadvantage is more evident for monthly simulation because the integration of the Lanczos' states using equation (6) is more demanding.

The RLANRM is more efficient than classical FD to simulate groundwater flow in complex conjunctive use systems, but its performance is slightly lower than the exhibited by the EVM. It also allows to deal with very large groundwater flow models, when many cells are used in the FD representation of the aquifer's spatial domain. Consequently, it is possible to consider more detailed descriptions on the spatial variability of the aquifer's hydraulic parameters and the imposed EA.

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